PRACTICE PROBLEMS FOR EXAM 2

Math 3160Q – Spring 2015
Professor Hohn

Below is a list of practice questions for Exam 2. Any quiz, homework, or example problem has a chance of being on the exam. For more practice, I suggest you work through the review questions at the end of each chapter as well.

1. Let X be a continuous random variable with density

$$f_X(t) = \begin{cases} 0 & t < -1 \\ t & -1 \leq t < 0 \\ ae^{-bt} & t \geq 0 \end{cases}$$

   and expected value 1.

   (a) What are a and b?

   (b) Find $\mathrm{Var}(X)$.

2. Suppose $X = N(\mu, \sigma^2)$. In terms of the distribution $\Phi(x) = P(N(0,1) \leq x)$ of the standard normal random variable, find the probability that X is less than $\frac{1}{2}\sigma + \mu$, or greater than $\frac{3}{2}\sigma + \mu$. That is, find $P\big((X < \frac{1}{2}\sigma + \mu) \cup (X > \frac{3}{2}\sigma + \mu)\big)$.

3. Suppose that an experiment has two outcomes 0 or 1 (such as flipping a coin). Suppose that you run n independent experiments and for the i^{th} experiment you let the random variable X_i tell you the outcome for $1 \leq i \leq n$. Then we can assume that for each i, that $X_i = \mathrm{Ber}(p)$ with $p = P(X_i = 1)$ (where we will assume for this problem that p is the same for each i). Then, let $X = \sum_{i=1}^{n} X_i$.

   (a) What is the state space S_X of X?

   (b) What is $\mathbb{E}[X]$?

4. Suppose that the time between customer arrivals in a store is given by an exponential random variable $X \stackrel{d}{=} \mathrm{Exp}(\lambda)$, such that the average time between arrivals is 2 minutes. Suppose you walk past the store and notice it's empty. What is the probability from the time you walk past the store, the store remains empty for more than 5 minutes?

5. Let X and Y be random variables with distributions given by,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 3x & 0 \le x < 1/3 \\ 1 & x \ge 1/3 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{2}e^{-2x} & x \ge 0 \end{cases}$$

(a) Find $P(X \le 1/4)$, $P(Y < 0)$, and $P(Y \le 0)$.

(b) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(c) Find $\mathbb{E}[Y]$.

6. Let X be a continuous random variable with density given by,

$$f_X(x) = \begin{cases} ke^x & 0 < x < \ln(2) \\ 0 & \text{otherwise} \end{cases}$$

(a) Find k.

(b) Let $Y = e^X$. Find the density $f_Y(y)$ of Y.

(c) What type of continuous random variable is Y?

7. You can't stand your probability professor. The amount of time you can sit in her classroom before storming out is modeled by an exponential random variable, where on average you storm out after 10 minutes.

(a) Given that you've already been sitting in her classroom for 30 minutes, what is the probability you will still be sitting in the classroom for 5 more minutes?

(b) What is the probability you are going to storm out in the next 10 minutes?

8. You are choosing between two venues to order food from which will be delivered to your house. With probability $1/3$ you will choose venue A, and with probability $2/3$ you will choose venue B. If you order from venue A, 15 minutes after making the call the remaining time it takes the food to arrive is exponentially distributed with average 10 minutes. If you order from venue B, 10 minutes after making the call, the time it takes the food will arrive is exponentially distributed with an average of 12 minutes. Given that you have already waited 25 minutes after calling, and the food has not arrived, what is the probability that you ordered from venue A?

9. Distracted while listening to the latest Beyoncé album, General Xavier accidentally knocks over a large jar filled with 10,000 fair coins at Fort Knox. All the coins fall out completely randomly. Let X count the number of heads that appear when the coins fall. Then $X \overset{d}{=} \mathrm{Bin}(10\,000, \frac{1}{2})$.

(a) (3 points) What is $P(X > 5100)$? You do not need to evaluate the sum you write down.

(b) (5 points) Approximate $P(X > 5100)$ using a normal distribution. Leave you answer in terms of $\Phi(x)$ where $\Phi(x) = P(N(0,1) \le x)$.

(c) (5 points) Approximate $P(X > 5100)$ using a Poisson distribution. Leaving an infinite sum here is OK.

10. 48000 fair dice are rolled independently. Let X count the number of sixes that appear.
    (a) What type of random variable is X?
    (b) Write the expression for the probability that between 7500 and 8500 sixes show. That is $P(7500 \leq X \leq 8500)$.
    (c) The sum you wrote in part b) is ridiculous to evaluate. Instead, approximate the value by a normal distribution and evaluate in terms of the distribution $\Phi(x) = P(N(0,1) \leq x)$ of a standard normal random variable.
    (d) Why do you think a normal distribution is a good choice for approximation?

11. Suppose that on average 2 people in a major city die each year from alien attack. Suppose that each attack is random and independent.
    (a) If X is the number of deaths from alien attack within the next year from a randomly selected major city, what type of random variable is X?
    (b) Use the Poisson approximation to approximate the probability that the next major city you visit will have at least 3 deaths due to alien attack?
    (c) Why do you think a Poisson approximation is used instead of a normal approximation?

12. Consider the following graph of the distribution $F_X(t)$ of X
    defined by
$$F_X(t) = \begin{cases} 0 & t < 0 \\ .3 & 0 \leq t < 1 \\ .8 & 1 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$$
    (a) Is the random variable X discrete, continuous, or neither?
    (b) What is the state space S_X of X?
    (c) What is the expected value $\mathbb{E}[X]$?

13. You have a fair coin, and you want to take your professor's money. You ask the professor to play a gambling game with you. The gambling game is designed as follows: You charge the professor \$$C$ to play. You then flip the coin twice and record the number of heads that show. If 0 heads show, you pay the professor \$5. If exactly 1 head shows, you pay the professor \$2. If 2 heads show, the professor pays you \$6. Let W be the random variable representing your wealth during a play of the game.
    (a) What elements are in the state space S_W of W?
    (b) What is the least amount of money \$$C$ you should charge your professor so that on average you don't lose money?

14. Suppose that X is a normal random variable with mean 75. Suppose that you know $\text{Var}(\frac{1}{2}X + 42) = 25$. Calculate $P(X < 60)$. You can leave your answer in terms of Φ, the CDF of a standard normal.

15. Suppose that you have a biased coin and two biased dice. The coin has a 60% chance of landing heads; die A has a 20% chance of rolling 3 and is equally likely to roll the other five faces; die B has a 30% chance of rolling 1, a 30% chance of rolling 2, and is equally likely to roll the other four faces. You play a "game" where you first flip the coin. If the coin lands on heads, you roll

die A and record the result. If, on the other hand, you flipped tails, then you roll die B and record the result. Let X be the outcome of the flipped coin ($\{X = 0\}$ is the event you flipped tails and $\{X = 1\}$ is the event you flipped heads), and let Y be the number you record when the die is rolled.

(a) Find the joint probability mass function $p_{X,Y}$ of X and Y.

(b) Find the marginal probability mass function p_Y of Y.

(c) Calculate $\mathbb{E}[Y]$.