Abstract. There have been many articles in the MONTHLY on quotient sets over the years. We take a first step here into the $p$-adic setting, which we hope will spur further research. We show that the set of quotients of nonzero Fibonacci numbers is dense in the $p$-adic numbers for every prime $p$.

The $n$th Fibonacci number $F_n$ is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Let $\mathbb{F} = \{1, 2, 3, 5, 8, 13, 21, \ldots\}$ denote the set of nonzero Fibonacci numbers and let $R(\mathbb{F}) = \{F_m/F_n : m, n \in \mathbb{N}\}$ be the set of all quotients of elements of $\mathbb{F}$. Binet’s formula

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - \tilde{\varphi}^n),$$

in which

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618 \ldots \text{ and } \tilde{\varphi} = \frac{1 - \sqrt{5}}{2} = -0.618 \ldots,$$

implies that $|F_n - \varphi^n/\sqrt{5}|$ tends to zero exponentially fast as $n \to \infty$ [37, p.57]. So as a subset of $\mathbb{R}$, $R(\mathbb{F})$ accumulates only at the points $\varphi^k$ for $k \in \mathbb{Z}$ [8, Ex.17].

On the other hand, $R(\mathbb{F})$ is a subset of the rational number system $\mathbb{Q}$, which can be endowed with metrics other than the one inherited from $\mathbb{R}$. For each prime $p$, there is a $p$-adic metric on $\mathbb{Q}$, with respect to which $\mathbb{Q}$ can be completed to form the set $\mathbb{Q}_p$ of $p$-adic numbers.

Our aim here is to prove the following theorem.

**Theorem 2.** $R(\mathbb{F})$ is dense in $\mathbb{Q}_p$ for every prime $p$.

By this we mean that the closure of $R(\mathbb{F})$ in $\mathbb{Q}_p$ with respect to the $p$-adic metric is $\mathbb{Q}_p$. Although there have been many papers in the MONTHLY on quotient sets over the years [1, 4, 8, 9, 15, 17, 26, 33], we are unaware of similar work being undertaken in the $p$-adic setting. We hope that this result will spur further investigations.

**Definition 3.** If $p$ is a prime number, then each $r \in \mathbb{Q}\setminus\{0\}$ has a unique representation

$$r = \pm p^k \frac{a}{b},$$

in which $k \in \mathbb{Z}$, $a, b \in \mathbb{N}$, and $a, b, p$ are pairwise relatively prime. The $p$-adic valuation $v_p : \mathbb{Q} \to \mathbb{Z}$ is defined by $v_p(0) = +\infty$ and, for $r$ as in (4), by $v_p(r) = k$. It satisfies $v_p(xy) = v_p(x) + v_p(y)$ for $x, y \in \mathbb{Q}$. The $p$-adic absolute value on $\mathbb{Q}$ is $\|r\|_p = p^{-v_p(r)}$ and the $p$-adic metric on $\mathbb{Q}$ is $d_p(x, y) = \|x - y\|_p$. 

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The \textit{\(p\)-adic number system} \(\mathbb{Q}_p\) is the completion of \(\mathbb{Q}\) with respect to the \(p\)-adic metric. Ostrowski’s theorem asserts that \(\mathbb{Q}_p\), for \(p\) prime, and \(\mathbb{R}\) are essentially the only completions of \(\mathbb{Q}\) that arise from an absolute value [20]. Thus, it is natural to seek results like Theorem 2, since similar questions in \(\mathbb{R}\) have long since been explored.

In fact, \(\mathbb{Q}_p\) is a field that contains \(\mathbb{Q}\) as a subfield. Its elements can be represented as infinite series, convergent with respect to the \(p\)-adic metric, of the form \(\sum_{n=0}^{\infty} a_n p^n\), in which \(k \in \mathbb{Z}\) and \(a_n \in \{0, 1, \ldots, p - 1\}\). The \(p\)-adic valuation, norm, and metric extend uniquely to \(\mathbb{Q}_p\) in a manner that respects these series representations.

\begin{example}
In \(\mathbb{Q}_2\) we have \(1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = -1\) since
\[
\left\| \sum_{n=0}^{N-1} 2^n - (-1) \right\|_2 = \left\| \frac{1 - 2^N}{1 - 2} + 1 \right\|_2 = \left\| 1 - (1 - 2^N) \right\|_2 = 2^N \leq 2^{-N}
\]
tends to zero as \(N \to \infty\).
\end{example}

Manipulating \(p\)-adic numbers is analogous to handling decimal expansions of real numbers. Instead of powers \(10^n\) with \(n\) running from some nonnegative \(N\) to \(-\infty\), we have powers \(p^n\) with \(n\) running from \(N\) (potentially negative) to \(+\infty\). Further details on \(\mathbb{Q}_p\) can be found in [11, 20].

To prove our theorem, we require a few preliminary results. In what follows, we write \(a|b\) to denote that the integer \(b\) is divisible by the integer \(a\). Proofs of the following assertions can be found in [37, p.82, p.73, & p.81].

\begin{lemma}
\begin{enumerate}[(a)]
\item If \(j|k\), then \(F_j|F_k\).
\item For each \(m \in \mathbb{N}\), there is a smallest index \(z(m)\) so that \(m|F_{z(m)}\) for all positive integers \(k\).
\item \(v_p(F_{z(p^n)}) = v_p(F_{z(p)}) + j\) for any odd prime \(p\) and \(j \in \mathbb{N}\).
\end{enumerate}
\end{lemma}

We also need a convenient dense subset of \(\mathbb{Q}_p\). We freely make use of the fact that \(d_p(x, y) \leq 1/p^k\) if and only if \(v_p(x - y) \geq k\).

\begin{lemma}
For each prime \(p\), the set \(\{p^{-n} : n, r \in \mathbb{N}\}\) is dense in \(\mathbb{Q}_p\).
\end{lemma}

\begin{proof}
Since \(\mathbb{Q}_p\) is the closure of \(\mathbb{Q}\) with respect to the \(p\)-adic metric, it suffices to show that each rational number can be arbitrarily well approximated, in the \(p\)-adic metric, by rational numbers of the form \(p^{-n}\) with \(n, r \in \mathbb{N}\). If \(x = p^k \sum_{j=0}^{\infty} a_j p^j \in \mathbb{Q}_p\), then let \(N \geq k + 1\) and \(n = \sum_{j=k}^{\infty} a_j p^j\) so that
\[
v_p(x - p^k n) = v_p(p^k \sum_{j=N-k}^{\infty} a_j p^j) = k + v_p(\sum_{j=N-k}^{\infty} a_j p^j) \geq k + (N - k) = N.
\]

An algebraic integer is a root of a monic polynomial with integer coefficients. For instance, \(2, \sqrt{5},\) and \(\varphi\) are algebraic integers. They are roots of \(x - 2, x^2 - 5,\) and \(x^2 - x - 1,\) respectively. The following is [24, Cor.1, p.15].

\begin{lemma}
a rational algebraic integer is an integer.
\end{lemma}
We also require a few facts about arithmetic in the field \( \mathbb{K} = \mathbb{Q}(\sqrt{5}) \). Let \( \mathcal{O}_\mathbb{K} \) denote the set of algebraic integers in \( \mathbb{K} \). It is a ring; in fact, \( \mathcal{O}_\mathbb{K} = \{a + b\varphi : a, b \in \mathbb{Z}\} \) [24, Cor.2, p.15]. In particular, \( \varphi, \overline{\varphi}, \) and \( \sqrt{5} \) each belong to \( \mathcal{O}_\mathbb{K} \).

An ideal in \( \mathcal{O}_\mathbb{K} \) is an additive subgroup \( i \) of \( \mathcal{O}_\mathbb{K} \) so that \( ai \subseteq i \) for all \( a \in \mathcal{O}_\mathbb{K} \). A prime in \( \mathcal{O}_\mathbb{K} \) refers to a prime ideal in the ring \( \mathcal{O}_\mathbb{K} \). A prime ideal is an ideal \( p \subseteq \mathcal{O}_\mathbb{K} \) with the property that, for any \( \alpha, \beta \in \mathcal{O}_\mathbb{K} \), the condition \( \alpha \beta \in p \) implies that \( \alpha \in p \) or \( \beta \in p \). To avoid confusion, we refer to the primes 2, 3, 5, 7, . . . as rational primes.

The product of two ideals \( i, j \) in \( \mathcal{O}_\mathbb{K} \) is defined by \( ij = \{\alpha\beta : \alpha \in i, \beta \in j\} \); it is also an ideal in \( \mathcal{O}_\mathbb{K} \). Positive powers of ideals are defined inductively by \( i^n = i(i^{n-1}) \) for \( n = 2, 3, . . . \). We say that \( i \) divides \( j \) if \( i \subseteq j \).

For an ideal \( i \) in \( \mathcal{O}_\mathbb{K} \), we write \( \alpha \equiv \beta \pmod{i} \) to mean that \( \beta - \alpha \in i \). The familiar properties of congruences hold when working modulo an ideal. For instance, \( \alpha \equiv \beta \pmod{i} \) implies that \( \alpha^j \equiv \beta^j \pmod{i} \) for \( j \in \mathbb{N} \). Similarly, if \( p \) and \( q \) are distinct prime ideals, \( \alpha \equiv \beta \pmod{p} \) and \( \alpha \equiv \beta \pmod{q} \), then \( \alpha \equiv \beta \pmod{pq} \).

**Lemma 9.** Let \( a, b \in \mathbb{Z} \) and \( j \in \mathbb{N} \). If \( a \equiv b \pmod{p^j} \), then \( p^j \) divides \( b - a \) in \( \mathbb{Z} \).

**Proof.** Suppose that \( a, b \in \mathbb{Z} \) and \( a \equiv b \pmod{p^j} \). Then, \( b - a = p^j\alpha \) for some \( \alpha \in \mathcal{O}_\mathbb{K} \). Since \( a = (b - a)/p^j \) is a rational algebraic integer, it is an integer by Lemma 8. Consequently, \( p^j \) divides \( b - a \) in \( \mathbb{Z} \).

Each rational prime \( p \) gives rise to an ideal \( \mathfrak{p} = p\mathcal{O}_\mathbb{K} \) in \( \mathcal{O}_\mathbb{K} \). This ideal factors uniquely as a product of prime ideals [24, Th.16, p.59]. In fact, since \( \mathbb{Q}(\sqrt{5}) \) is a quadratic extension of \( \mathbb{Q} \) we have [24, p.74]:

**Lemma 10.** Let \( \mathbb{K} = \mathbb{Q}(\sqrt{5}) \) and let \( p \) be a rational prime. Then \( \mathfrak{p} = p\mathcal{O}_\mathbb{K} \) is the product of at most two prime ideals (not necessarily distinct).

We can be a little more specific. The only rational prime \( p \) for which \( p \) is a square of a prime ideal is \( p = 5 \); this reflects the factorization \( 5 = (\sqrt{5})^2 \) [24, Th.24, p.72].

**Proof of Theorem 2.** There are two cases, according to whether \( p \) is odd or \( p = 2 \). Let \( p \) be an odd rational prime and let \( \mathfrak{p} = p\mathcal{O}_\mathbb{K} \). For each \( j \in \mathbb{N} \), Lemma 6 ensures that \( p^{2j} | F_{z(p)p^{2j}} \). Since \( \sqrt{5} \in \mathcal{O}_\mathbb{K} \), (1) reveals that

\[
\varphi^{z(p)p^{2j}} - \overline{\varphi}^{z(p)p^{2j}} = \sqrt{5}F_{z(p)p^{2j}} \equiv 0 \pmod{p^{2j}},
\]

and hence

\[
\varphi^{z(p)p^{2j}} \equiv \overline{\varphi}^{z(p)p^{2j}} \pmod{q^{2j}}
\]

for each prime \( q \) in \( \mathcal{O}_\mathbb{K} \) that divides \( \mathfrak{p} \). Since \( \overline{\varphi} = -1/\varphi \), we have

\[
\varphi^{2z(p)p^{2j}} \equiv \varphi^{-2z(p)p^{2j}} \pmod{q^{2j}},
\]

so that

\[
\varphi^{4z(p)p^{2j}} \equiv 1 \pmod{q^{2j}} \quad \text{and} \quad \varphi^{4z(p)p^{2j}} \equiv 1 \pmod{q^{2j}}.
\]

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Setting \( x = \varphi^{4z(p)p^j} \) and \( y = \varphi^{4z(p)p^j} \) in the identity
\[
x^m - y^m = (x - y)(x^{m-1} + x^{m-2}y + \cdots + y^{m-1}), \quad m \in \mathbb{N},
\]
implies that
\[
\frac{F_{4z(p)p^j/m}}{F_{4z(p)p^j}} = \frac{\varphi^{4z(p)p^j/m} - \varphi^{4z(p)p^j}}{\varphi^{4z(p)p^j} - \varphi^{4z(p)p^j}} = (\varphi^{4z(p)p^j})^{m-1} + (\varphi^{4z(p)p^j})^{m-2}(\varphi^{4z(p)p^j}) + \cdots + (\varphi^{4z(p)p^j})^{m-1}
\equiv m \pmod{p^j}
\]
by (11). Now Lemma 10 ensures that
\[
\frac{F_{4z(p)p^{2j/m}}}{F_{4z(p)p^{2j}}} \equiv m \pmod{p^j}
\]
for any \( m \in \mathbb{N} \). However, \( F_{4z(p)p^{2j/m}}/F_{4z(p)p^{2j}} \) is a rational integer by Lemma 6, so
\[
\frac{F_{4z(p)p^{2j/m}}}{F_{4z(p)p^{2j}}} \equiv m \pmod{p^j} \quad (12)
\]
by Lemma 9. Appealing to Lemma 6, we have
\[
v_p(F_{4z(p)p^{2(j+2r)}}) = v_p(F_{4z(p)p^{2(k+r)}}) + 2r, \quad (13)
\]
so
\[
\frac{F_{4z(p)p^{2(k+2r)}}}{F_{4z(p)p^{2(k+r)}}} = p^{2r} \ell, \quad \ell \in \mathbb{Z}, \quad \gcd(\ell, p) = 1. \quad (14)
\]
Set \( m = p'n'\ell \) and \( j = k + r \) in (12) and use (14) to conclude that
\[
p^{2r} \ell \cdot \frac{F_{4z(p)p^{2(k+r)}}}{F_{4z(p)p^{2(k+2r)}}} = \frac{F_{4z(p)p^{2(k+r)}}}{F_{4z(p)p^{2(k+2r)}}} \equiv \frac{p'n'\ell}{p^{k+r}} \pmod{p^{k+r}}.
\]
Since \( \gcd(\ell, p) = 1 \), the preceding yields
\[
p^{2r} \ell \cdot \frac{F_{4z(p)p^{2(k+3r)n\ell}}}{F_{4z(p)p^{2(k+4r)}}} \equiv n \pmod{p^k},
\]
and hence
\[
v_p \left( \frac{F_{4z(p)p^{2k+3r}n\ell}}{F_{4z(p)p^{2k+4r}}} - p^{-r}n \right) \geq k.
\]
This holds for all \( n, r \in \mathbb{N} \) and all \( k > r \), so \( R(F) \) is dense in \( \mathbb{Q}_p \) by Lemma 7.

If \( p = 2 \) we must replace the exponents \( 4z(p)p^j \) in (11) with \( 3 \cdot 2^{j+2} \) since \( z(2) = 3 \) and 4 = 2². The proof can proceed as before if we replace (13) with the corresponding statement for \( p = 2 \). It suffices to show that \( v_2(F_{3/2}) = j + 2 \) for \( j \in \mathbb{N} \); see [22] for
a more general result. We proceed by induction on \( j \). The base case \( j = 1 \) is \( v_2(F_0) = v_2(8) = 3 \). Now suppose that \( v_2(F_{3j/2}) = j + 2 \) for some \( j \geq 2 \). Let \( L_n \) denote the \( n \)th Lucas number; these satisfy the recurrence \( L_{n+2} = L_{n+1} + L_n \) with initial conditions \( L_0 = 2 \) and \( L_1 = 1 \). Since \( L_{2n} = 2(-1)^n + 5F_n^2 \) [37, p.177] and since \( 2 = F_3 \) divides \( F_{3j/2-1} \), we have

\[
v_2(L_{3j/2}) = v(L_{2(3j/2-1)}) = v_2(2(-1)^{3j/2-1} + 5F_{3j/2-1}^2) = 1.
\]

Because \( F_{2n} = F_n L_n \) [37, p.177], we conclude that

\[
v_2(F_{3j/2+1}) = v_2(F_{2(3j/2)}) = v_2(F_{3j/2}) + v_2(L_{3j/2}) = (j + 2) + 1 = (j + 1) + 2,
\]

which completes the induction.

\[\blacksquare\]

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