

Which Weighted Composition Operators are Complex Symmetric?

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Abstract. Recent work by several authors has revealed the existence of many unexpected classes of normal weighted composition operators. On the other hand, it is known that every normal operator is a complex symmetric operator. We therefore undertake the study of complex symmetric weighted composition operators, identifying several new classes of such operators.

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1. Introduction

In 2010, C. Cowen and E. Ko obtained an explicit characterization and spectral description of all Hermitian weighted composition operators on the classical Hardy space H^2 [5]. This work was later extended to certain weighted Hardy spaces by C. Cowen, G. Gunatillake, and E. Ko [4]. Along similar lines, P. Bourdon and S. Narayan have recently studied normal weighted composition operators on H^2 [1]. Taken together, these articles have established the existence of several unexpected families of normal weighted composition operators.

It turns out that normal operators are the simplest examples of complex symmetric operators. We say that a bounded operator T on a complex Hilbert space \mathcal{H} is *complex symmetric* if there exists a *conjugation* (i.e., a conjugate-linear, isometric involution) J such that $T = JT^*J$. The general study of such operators was undertaken by the first author, M. Putinar, and W. Wogen, in various combinations, in [7–10]. A number of other authors have also made significant contributions [3, 11–14, 17–20].

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We consider here the problem of describing all complex symmetric weighted composition operators. Among other results, we produce a class of complex symmetric weighted composition operators which includes the Hermitian examples obtained in [4, 5] as special cases. We also raise a number of open questions which we hope will spur further research.

2. Observations and results

In what follows, we let $H^2(\beta)$ denote the weighted Hardy space which corresponds to the weight sequence $\{\beta(n)\}_{n=0}^{\infty}$ [6, Sect. 2.1]. For each w in the open unit disk \mathbb{D} and every integer $n \geq 0$, we let $K_w^{(n)}$ denote the unique function in $H^2(\beta)$ which satisfies $\langle f, K_w^{(n)} \rangle = f^{(n)}(w)$ for every f in $H^2(\beta)$. For convenience, we often choose to write K_w in place of $K_w^{(0)}$. If $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is analytic, then the *composition operator* $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is defined by setting

$$C_\varphi(f) = f \circ \varphi.$$

Given another analytic function $\psi : \mathbb{D} \rightarrow \mathbb{C}$, we define the *weighted composition operator* $W_{\varphi,\psi}$ by setting

$$W_{\varphi,\psi}(f) = \psi \cdot (f \circ \varphi).$$

Assuming that $W_{\varphi,\psi}$ is bounded, one has the useful formulae

$$W_{\varphi,\psi}^*(K_w) = \overline{\psi(w)} K_{\varphi(w)}, \quad (1)$$

$$W_{\varphi,\psi}^*(K_w^{(1)}) = \overline{\psi(w)} \overline{\varphi'(w)} K_{\varphi(w)}^{(1)} + \overline{\psi'(w)} K_{\varphi(w)}. \quad (2)$$

2.1. Composition operators

One initially expects few unweighted composition operators to be complex symmetric. In fact, the only obvious candidates which come to mind are the normal composition operators. These are precisely the operators $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ where $\varphi(z) = az$ and $|a| \leq 1$ [6, Thm. 8.2]. One might initially suspect that these are the *only* complex symmetric composition operators. This naïve conjecture proves to be false, however, as there exist at least two other basic families of complex symmetric composition operators.

Proposition 2.1. *If φ is either (i) constant, or (ii) an involutive disk automorphism, then $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is a complex symmetric operator.*

The preceding follows immediately from the fact that an operator which is algebraic of degree two is complex symmetric [10, Thm. 2]. In what follows, we work only with nonconstant symbols φ . It turns out that (ii) prompts an elementary question whose answer has so far eluded us.

Question 1. Let φ be an involutive disk automorphism. Find an explicit conjugation $J : H^2(\beta) \rightarrow H^2(\beta)$ such that $C_\varphi = JC_\varphi^*J$.

Naturally, one is also interested in determining whether there are any additional classes of complex symmetric composition operators.

Question 2. Characterize all complex symmetric composition operators C_φ on the classical Hardy space H^2 or, more generally, on weighted Hardy spaces $H^2(\beta)$.

In the negative direction, we have the following results.

Proposition 2.2. *If $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is a hyponormal composition operator which is complex symmetric, then $\varphi(z) = az$ where $|a| \leq 1$.*

Proof. Suppose C_φ is hyponormal; that is, $\|C_\varphi f\| \geq \|C_\varphi^* f\|$ for all f in $H^2(\beta)$. If C_φ is J -symmetric, then it follows that

$$\|C_\varphi^* f\| = \|JC_\varphi Jf\| = \|C_\varphi Jf\| \geq \|C_\varphi^* Jf\| = \|JC_\varphi f\| = \|C_\varphi f\|.$$

Thus $\|C_\varphi f\| = \|C_\varphi^* f\|$ for all f in H^2 whence C_φ is normal. By [6, Thm. 8.2] we conclude that $\varphi(z) = az$ where $|a| \leq 1$. \square

Proposition 2.3. *Suppose that $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is J -symmetric. If $J(1)$ is a constant multiple of a kernel function K_w , then $\varphi(w) = w$. The converse holds whenever φ is not an automorphism.*

Proof. If $J(1) = \gamma K_w$ for some constant $\gamma \neq 0$ and C_φ is J -symmetric, then

$$\gamma K_w = J(1) = JC_\varphi(1) = C_\varphi^* J(1) = C_\varphi^*(\gamma K_w) = \gamma K_{\varphi(w)},$$

from which we conclude that $\varphi(w) = w$. On the other hand, suppose that $\varphi(w) = w$. Since $C_\varphi^*(K_w) = K_{\varphi(w)} = K_w$, we see that

$$C_\varphi J(K_w) = JC_\varphi^*(K_w) = J(K_w).$$

As long as φ is not an automorphism, the only eigenvectors for C_φ corresponding to the eigenvalue 1 are the constant functions [16, p. 90]. Therefore $J(K_w)$ must be a constant function, which means that $J(1)$ must be a scalar multiple of K_w . \square

In light of the preceding, we see that if J is a conjugation on $H^2(\beta)$ such that $J(1)$ is not a constant multiple of a kernel function, then there does not exist a J -symmetric composition operator C_φ on $H^2(\beta)$ whose symbol fixes a point in \mathbb{D} . If $J(1)$ is a constant multiple of 1, then we can say even more about φ . The following is inspired by an unpublished result of P. Bourdon and D. Szajda [6, Ex. 8.1.2].

Proposition 2.4. *Suppose that $J : H^2(\beta) \rightarrow H^2(\beta)$ is a conjugation, $J(1)$ is a constant multiple of 1, and $J(z)$ is a constant multiple of z^m for some $m \geq 1$. If C_φ is J -symmetric, then $\varphi(z) = az$ for some $|a| \leq 1$.*

Proof. Since $1 = \beta(0)K_0$, it follows from Proposition 2.3 that $\varphi(0) = 0$, whence

$$C_\varphi^*(K_0^{(1)}) = \overline{\varphi'(0)}K_{\varphi(0)}^{(1)} = \overline{\varphi'(0)}K_0^{(1)}$$

by (2). Thus $z = \beta(1)K_0^{(1)}$ is an eigenvector for C_φ^* corresponding to the eigenvalue $\overline{\varphi'(0)}$. Since C_φ is J -symmetric, z^m must be an eigenvector for C_φ corresponding to the eigenvalue $\varphi'(0)$. Observe that $C_\varphi(z^m) = \varphi^m$, which means that $\varphi(z)^m = \varphi'(0)z^m$. Consequently $\varphi(z) = az$, where $|a| \leq 1$. \square

2.2. Weighted composition operators

Although our list of complex symmetric composition operators is somewhat sparse, there are a variety of *weighted* composition operators which are known to be complex symmetric. Indeed, the study of Hermitian, normal, and unitary weighted composition operators has been the focus of intense research [1, 4, 5]. The following is a generalization of [1, Lem. 2, Prop. 3], where the same conclusion is obtained under the assumption that $W_{\varphi,\psi}$ is normal.

Proposition 2.5. *If $W_{\varphi,\psi} : H^2(\beta) \rightarrow H^2(\beta)$ is complex symmetric, then either ψ is identically zero or ψ is nonvanishing on \mathbb{D} . Moreover, if φ is not a constant function and ψ is not identically zero, then φ is univalent.*

Proof. Suppose that $W_{\varphi,\psi}$ is complex symmetric and that ψ does not vanish identically. Since $\ker W_{\varphi,\psi} = \{0\}$, we conclude that $\ker W_{\varphi,\psi}^* = \{0\}$ by [7, Prop. 1]. If $\psi(w) = 0$ for some w in \mathbb{D} , then $W_{\varphi,\psi}^*(K_w) = 0$ by (1). Since this contradicts the fact that $\ker W_{\varphi,\psi}^*$ is trivial, we conclude that ψ is nonvanishing on \mathbb{D} . Now suppose that there are points w_1 and w_2 in \mathbb{D} such that $\varphi(w_1) = \varphi(w_2)$. It follows that

$$W^*(\overline{\psi(w_2)}K_{w_1} - \overline{\psi(w_1)}K_{w_2}) = \overline{\psi(w_2)}\overline{\psi(w_1)}K_{\varphi(w_1)} - \overline{\psi(w_1)}\overline{\psi(w_2)}K_{\varphi(w_2)} = 0.$$

Since any distinct pair of reproducing kernel functions is linearly independent, we conclude that $w_1 = w_2$. In other words, φ is univalent. □

The following result provides a severe restriction on the spectrum of a complex symmetric weighted composition operator whose symbol has a fixed point in \mathbb{D} .

Proposition 2.6. *Suppose that $W_{\varphi,\psi} : H^2(\beta) \rightarrow H^2(\beta)$ is a complex symmetric operator. If $\varphi(w_0) = w_0$ for some w_0 in \mathbb{D} , then $\psi(w_0) \varphi'(w_0)^n$ is an eigenvalue of $W_{\varphi,\psi}$ for every integer $n \geq 0$.*

Proof. Since $W_{\varphi,\psi}$ is complex symmetric, by [7, Prop. 1] it suffices to prove that

$$\overline{\psi(w_0) \varphi'(w_0)^n} \tag{3}$$

is an eigenvalue for $W_{\varphi,\psi}^*$. Let us first assume that $\varphi'(w_0)$ is not a root of unity.

We claim that for each $n \geq 0$, the function $K_{w_0}^{(n)}$ can be written in the form

$$v_n + \alpha_{n-1}v_{n-1} + \alpha_{n-2}v_{n-2} + \cdots + \alpha_0v_0,$$

where v_j is an eigenvector for $W_{\varphi,\psi}$ corresponding to the eigenvalue $\overline{\psi(w_0) \varphi'(w_0)^j}$. We prove this assertion by induction. Note that

$$W_{\varphi,\psi}^*(K_{w_0}) = \overline{\psi(w_0)}K_{\varphi(w_0)} = \overline{\psi(w_0)}K_{w_0},$$

so the claim holds when $n = 0$. Suppose then that the claim holds for all $n \leq k$ and consider the kernel function $K_{w_0}^{(k+1)}$. Now recall that $W_{\varphi,\psi}^*(K_{w_0}^{(k+1)})$ equals $\overline{\psi(w_0) \varphi'(w_0)^{k+1}}K_{\varphi(w_0)}^{(k+1)}$ plus a linear combination of kernel functions $K_{w_0}^{(j)}$ with

$j \leq k$. Our induction hypothesis implies that each of these kernel functions is a linear combination of eigenvectors v_j . Therefore we may write

$$W_{\varphi,\psi}^*(K_{w_0}^{(k+1)}) = \overline{\psi(w_0)} \overline{\varphi'(w_0)^{k+1}} K_{w_0}^{(k+1)} + \beta_k v_k + \beta_{k-1} v_{k-1} + \cdots + \beta_0 v_0$$

for some constants $\beta_0, \beta_1, \dots, \beta_k$. Observe that the function

$$v_{k+1} = K_{w_0}^{(k+1)} + \sum_{j=0}^k \frac{\beta_j}{\overline{\psi(w_0)} (\overline{\varphi'(w_0)^{k+1}} - \overline{\varphi'(w_0)^j})} v_j$$

is an eigenvector for $W_{\varphi,\psi}^*$ corresponding to the eigenvalue $\overline{\psi(w_0)} \overline{\varphi'(w_0)^{k+1}}$. Consequently our claim holds for all n . In other words, every term (3) is an eigenvalue for $W_{\varphi,\psi}^*$. If $\varphi'(w_0)$ is an m th root of unity, then a similar argument shows that

$$K_{w_0}^{(n)} = v_n + \alpha_{n-1} v_{n-1} + \alpha_{n-2} v_{n-2} + \cdots + \alpha_0 v_0$$

whenever $0 \leq n \leq m-1$. Hence (3) is an eigenvalue for $W_{\varphi,\psi}^*$ when $n \leq m-1$ and hence for all n . In either case, every number (3) is an eigenvalue for $W_{\varphi,\psi}^*$, which means that $\psi(w_0) \varphi'(w_0)^n$ is an eigenvalue for $W_{\varphi,\psi}$. \square

Example 1. Fix $a \in \mathbb{D} \setminus \{0\}$ and let

$$\varphi = \frac{a - z}{1 - \bar{a}z}.$$

Since φ is an involutive automorphism, the composition operator $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is complex symmetric by Proposition 2.1. Moreover, observe that the spectrum $\sigma(C_\varphi)$ of C_φ is precisely $\{-1, 1\}$. On the other hand, Proposition 2.6 implies that $\varphi'(w_0)^n$ belongs to $\sigma(C_\varphi)$ whenever w_0 is a fixed point of w_0 . However, the only fixed point of φ which lies inside of \mathbb{D} is

$$w_0 = \frac{1 - \sqrt{1 - |a|^2}}{\bar{a}},$$

which happens to satisfy $\varphi'(w_0) = -1$, in accordance with Proposition 2.6.

2.3. Koenigs eigenfunctions

For any nonconstant non-automorphism $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ which has a fixed point w_0 in \mathbb{D} and for which $\varphi'(w_0) \neq 0$, there is an analytic $\kappa : \mathbb{D} \rightarrow \mathbb{C}$ such that $\kappa \circ \varphi = \varphi'(w_0)\kappa$. This function, called the *Koenigs eigenfunction* for φ , is unique up to scalar multiplication [6, p. 62, p. 93]. Furthermore, κ^n (or any constant multiple thereof) is the only analytic function for which $\kappa^n \circ \varphi = \varphi'(w_0)^n \kappa^n$. Proposition 2.6, together with the details of its proof, yields the following result pertaining to unweighted composition operators.

Proposition 2.7. *Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic selfmap which is not an automorphism and suppose that $\varphi(w_0) = w_0$ and $\varphi'(w_0) \neq 0$ for some w_0 in \mathbb{D} . If $C_\varphi : H^2(\beta) \rightarrow H^2(\beta)$ is complex symmetric, then every power κ^n of the Koenigs eigenfunction for φ belongs to $H^2(\beta)$.*

It is not difficult to construct a univalent map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ in such a way that one can readily determine whether its Koenigs eigenfunction belongs to $H^2(\beta)$ [16, pp. 93-94]. Let $\kappa : \mathbb{D} \rightarrow \mathbb{C}$ be a univalent function that vanishes at some point w_0 and consider the region $\kappa(\mathbb{D})$. Suppose that $\lambda\kappa(\mathbb{D}) \subseteq \kappa(\mathbb{D})$ for some complex λ with $|\lambda| < 1$. Define the map φ by $\varphi(z) = \kappa^{-1}(\lambda\kappa(z))$. Then, by construction, φ is a univalent self-map of \mathbb{D} that fixes w_0 and whose Koenigs eigenfunction is κ . Hence, by starting with a κ that belongs to $H^2(\beta)$, we construct a φ whose Koenigs function belongs to $H^2(\beta)$. Similarly, if we take κ such that κ^n does not belong to $H^2(\beta)$ for some n , we obtain a map whose corresponding composition operator is not complex symmetric by Proposition 2.7. For example, consider any such λ and take $\kappa(z) = 2z/(1 - z)$, which does not belong to the Hardy space H^2 . From this we obtain the map $\varphi(z) = (\lambda z)/(1 + (\lambda - 1)z)$, which induces a composition operator $C_\varphi : H^2 \rightarrow H^2$ which is not complex symmetric.

Much work has been done to determine the conditions under which a Koenigs eigenfunction κ belongs to the Hardy space H^2 . In this context, Proposition 2.7 is equivalent to saying that κ belongs to H^p for every $0 < p < \infty$. The following proposition follows directly from [15, Thm. 2.2].

Proposition 2.8. *Suppose that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is not an automorphism and that φ has a fixed point w_0 in \mathbb{D} such that $\varphi'(w_0) \neq 0$. If $C_\varphi : H^2 \rightarrow H^2$ is complex symmetric, then the essential spectral radius of C_φ is 0. In other words, C_φ must be a Riesz composition operator.*

A good deal of work has been done to study Riesz composition operators on H^2 . Bourdon and Shapiro’s paper [2] serves as an excellent starting point.

Suppose that φ is not an automorphism, $\varphi(w_0) = w_0$, $\varphi'(w_0) \neq 0$, and that C_φ is J -symmetric. As we have already observed, $J(1)$ must be a constant multiple of K_{w_0} . Let κ denote the Koenigs eigenfunction for φ , normalized so that $\|\kappa\| = 1$. We also know that $J(\kappa)$ equals a constant multiple of $K_{w_0}^{(1)}$. In particular, taking into account the norms of these functions, we can write

$$J(1) = \frac{\gamma \beta(0) K_{w_0}}{\|K_{w_0}\|}, \quad J(\kappa) = \frac{\delta K_{w_0}^{(1)}}{\|K_{w_0}^{(1)}\|},$$

where $|\gamma| = |\delta| = 1$. Since $\langle \kappa, 1 \rangle = \langle J(1), J(\kappa) \rangle$, we see that

$$|\kappa(0)| = \frac{|K_{w_0}^{(1)}(w_0)|}{\|K_{w_0}\| \|K_{w_0}^{(1)}\|}.$$

If $w_0 = 0$, then this tells us nothing. If $w_0 \neq 0$, however, it places a major restriction upon the function κ . In essence, most functions in $H^2(\beta)$ cannot be Koenigs eigenfunctions for complex symmetric composition operators.

2.4. An instructive example

We conclude this note by producing a class of complex symmetric weighted composition operators which includes the Hermitian examples obtained in [4, 5] as

special cases. For each $\kappa \geq 1$, let $H^2(\beta_\kappa)$ denote the weighted Hardy space whose reproducing kernel is $K_w(z) = (1 - \bar{w}z)^{-\kappa}$. We now explicitly characterize all weighted composition operators on $H^2(\beta_\kappa)$ which are J -symmetric with respect to the conjugation

$$[Jf](z) = \overline{f(\bar{z})} \tag{4}$$

on $H^2(\beta_\kappa)$. For the sake of convenience, we sometimes write $\tilde{f} := Jf$.

Proposition 2.9. *A weighted composition operator $W_{\varphi,\psi} : H^2(\beta_\kappa) \rightarrow H^2(\beta_\kappa)$ is J -symmetric with respect to the conjugation (4) if and only if*

$$\psi(z) = \frac{b}{(1 - a_0z)^\kappa}, \quad \varphi(z) = a_0 + \frac{a_1z}{1 - a_0z}, \tag{5}$$

where a_0 and a_1 are constants such that φ maps \mathbb{D} into \mathbb{D} . Moreover, such an operator is normal if and only if either,

- (i) $b = 0$,
- (ii) $b \neq 0$ and $\text{Im } a_0\bar{a}_1 = (1 - |a_0|^2) \text{Im } a_0$.

Moreover, $W_{\varphi,\psi}$ is Hermitian if and only if a_0, a_1 , and b each belong to \mathbb{R} .

Proof. To streamline our notation, we let $W := W_{\varphi,\psi}$. A simple computation now confirms that if ψ and φ are given by (5), then $WJK_w = JW^*K_w$ for all w in \mathbb{D} , implying that $W = JW^*J$. On the other hand, if $W = JW^*J$, then $WJK_w = JW^*K_w$ for all w in \mathbb{D} . Since $JK_w = K_{\bar{w}}$, this implies that

$$\psi(z)K_{\bar{w}}(\varphi(z)) = \psi(w)K_{\overline{\varphi(w)}}(z) \tag{6}$$

holds for all z, w in \mathbb{D} . Setting $w = 0$ in the preceding we find that

$$\psi(z) = \frac{\psi(0)}{(1 - \varphi(0)z)^\kappa}.$$

Thus ψ is of the form (5) with $b = \psi(0)$ and $a_0 = \varphi(0)$. From (6) it follows that

$$\frac{1 - \varphi(w)z}{1 - a_0z} = \frac{1 - \varphi(z)w}{1 - a_0w}.$$

Writing $\varphi(z) = a_0 + z\xi(z)$ where ξ is analytic on \mathbb{D} , we see that

$$(1 - a_0z)\xi(z) = (1 - a_0w)\xi(w)$$

for all z, w in \mathbb{D} . Thus the function $(1 - a_0z)\xi(z)$ is constant. Letting $\xi(0) = a_1$, we conclude that φ has the form (5).

Suppose that ψ and φ are given by (5) and note that W is normal if and only if $JWW^*K_w = WW^*JK_w$ for all w in \mathbb{D} . The preceding condition is equivalent to asserting that

$$\frac{\psi(w)\tilde{\psi}(z)}{1 - \varphi(w)\tilde{\varphi}(z)} = \frac{\tilde{\psi}(w)\psi(z)}{1 - \tilde{\varphi}(w)\varphi(z)}$$

holds for all z, w in \mathbb{D} . Taking the reciprocal of the preceding and simplifying, we see that equality holds for all z, w if and only if either $b = 0$ or $b \neq 0$ and $\text{Im } a_0\bar{a}_1 = (1 - |a_0|^2) \text{Im } a_0$.

We also note that $W = W^*$ if and only if $WJK_w = JWK_w$, which yields

$$\psi(z)K_{\bar{w}}(\varphi(z)) = \tilde{\psi}(z)K_{\bar{w}}(\tilde{\varphi}(z)).$$

Setting $w = 0$ in the preceding yields $\psi(z) = \tilde{\psi}(z)$ so that a_0 and b are real. This implies that $\varphi(z) = \tilde{\varphi}(z)$ whence a_1 is also real. Conversely, it is easy to see that if a_0 , a_1 , and b are real, then W is Hermitian. \square

It follows from the preceding that if a_0, a_1, b are chosen so that φ maps \mathbb{D} into \mathbb{D} and so that (i) and (ii) both fail to hold, then the operator $W_{\varphi, \psi} : H^2(\beta_\kappa) \rightarrow H^2(\beta_\kappa)$ will be complex symmetric and non-normal. Moreover, the operators produced by Proposition 2.9 include the Hermitian examples considered in [4, 5].

Question 3. The detailed spectral structure of *Hermitian* weighted composition operators $W_{\varphi, \psi} : H^2(\beta_\kappa) \rightarrow H^2(\beta_\kappa)$ with ψ and φ given by (5) is studied in [4, 5]. What is the corresponding spectral theory for the non-normal weighted composition operators arising from Proposition 2.9?

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