Games of Chains and Cutsets in the Boolean Lattice

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Abstract

B. Bájnok and S. Shahriari proved that in $2^{[n]}$, the Boolean lattice of order n, the width (the maximum size of an antichain) of a non-trivial cutset (a collection of subsets that meets every maximal chain and does not contain $\emptyset$ or $[n]$) is at least $n-1$. We prove that, for $n \geq 5$, in the Boolean lattice of order n, given $\lceil \frac{n}{2} \rceil - 1$ disjoint long chains, we can enlarge the collection to a cutset of width $n-1$. However, there exists a collection of $\lceil \frac{n}{2} \rceil$ long chains that cannot be so extended.

Introduction and the result

Let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ be the partially ordered set of all the subsets of $[n]$ ordered by inclusion. In other words, $2^{[n]}$ is the Boolean lattice of order n. A *long* chain in $2^{[n]}$ is a collection of $n-1$ subsets $A_1 \subset A_2 \subset \cdots \subset A_{n-1}$ such that $|A_i| = i$. A long chain in $2^{[n]}$ is just a maximal chain minus the empty set and the full set. Let $\mathcal{C} \subseteq 2^{[n]}$. $\mathcal{C}$ is a *cutset* in $2^{[n]}$ if it intersects every long chain non-trivially. In other words, if we consider the Hasse diagram of $2^{[n]}$ as a graph, then a cutset is a collection of subsets whose removal would place the empty set and the full set in two different connected components. (Note that usually a cutset is defined to be a collection of subsets that meets every maximal (as opposed to long) chain [3, 4]. The disadvantage of the usual definition is that any collection of subsets that includes the empty set or the full set will (trivially) be a cutset.) Any collection of pairwise incomparable elements in a poset is called an *anti-chain*, and the size of the largest anti-chain in a poset is the *width* of the poset. Theorem 1 of [2] together with Dilworth's theorem (Theorem 3.2.1 of [1]) gives the following:

Proposition 1 ([2]) *Let $\mathcal{C}$ be a cutset in the Boolean lattice of order n, with $n \geq 2$. Then the width of $\mathcal{C}$ is greater or equal to $n-1$. Furthermore, for $n \geq 3$ there does exists cutsets of width $n-1$ in the Boolean lattice of order n.*

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The purpose of this paper is to further investigate the properties of cutsets in the Boolean lattice. We do this by introducing a game for two players on the Boolean lattice. By doing so the results of [2] will become a special case of the game and the results of this paper will be the obvious next case.

Let n, a_1, …, a_m be fixed positive integers with $a_1+a_2+\cdots+a_m \leq n$. The game $G(a_1, a_2, \ldots, a_m)$ for two players and with a maximum of m moves is defined as follows: For the first move player I will choose a_1 pairwise disjoint long chains in $2^{[n]}$. For the second move player II will extend the collection to $a_1 + a_2$ pairwise disjoint long chain by adding another a_2 such chains. The game continues by players taking turns and successively adding a_3, a_4, …, a_m disjoint long chains to the collection. The last player to complete a move wins.

As an example, consider the game $G(\underbrace{1, 1, \ldots, 1}_{n})$ for $n \geq 2$. In this game the players take turns to construct a long chain disjoint from the previous ones. It is easy to see that the second player has a winning strategy (regardless of whether n is even or odd!) The strategy is to always pick the long chain that is composed of the complements of the sets that the first player just used. Since the first player cannot use a set and its complement in any chain, and since the complements of the sets in a long chain do form another long chain, the second player will always be able to make a move and hence wins.

Note that it is easy to find n pairwise disjoint long chains in $2^{[n]}$ (for example, using a symmetric chain decomposition [1, Theorem 3.1.1]) and thus for $m = 1$ and $1 \leq a \leq n$ player I wins the (boring) game $G(a)$. For $m = 2$ the game becomes interesting and the complete solution follows from the results of [2] and is as follows:

Theorem 2 ([2]) *Let $n \geq 4$, and let a, b be positive integers with $a + b \leq n$. Define the game $G(a, b)$ as above. Then*

player I wins if $a = n - 1$ and $b = 1$,

player I wins if $a = n - 2$ and $b = 2$,

player II wins if $a = n - 2$ and $b = 1$,

player II wins if $1 \leq a \leq n - 3$.

In other words, for $n \geq 4$, in the Boolean lattice of order n, given $n - 3$ disjoint long chains one can always find 3 more such chains, whereas given $n - 2$ disjoint long chains there will always be one more such chain but not necessarily 2 more. The former has the more involved proof, and in fact instead of the latter it was proved in [2] that given any collection of $n - 2$ chains (not necessarily long nor necessarily disjoint) there exists at least one more long chain. This means that any collection of subsets that can be covered by $n - 2$ chains cannot be a cutset and hence (since, by Dilworth's theorem, the width of a poset is equal to the minimum number of chains needed to cover it) Proposition 1 follows.

From Theorem 2 we can conclude that for any game $G(a_1, a_2, \ldots, a_m)$, with $a_1+a_2+\cdots+a_m < n$ and $n \geq 4$, the last player who is scheduled to move will be able to complete her move and hence wins. Thus the interesting case is when $a_1 + \cdots + a_m = n$. It also follows from Theorem 2 that, for $n \geq 4$, if $a_m \geq 3$ then again the last player who is scheduled to move will win.

Consider the above game when $m = 3$. From the previous remarks the only unresolved cases are the games $G(a, b, c)$ where $a+b+c = n$ and $c = 1$ or 2. In this paper we will analyze the games $G(a, b, 1)$ with $a + b + 1 = n$ and prove the following:

Theorem 3 *Let n, a, b be positive integers with $a + b = n - 1$ and $n \geq 5$. Consider the game $G(a, b, 1)$. Then player I wins if and only if $a \geq \lceil \frac{n}{2} \rceil$.*

It follows that, for $n \geq 5$, in the general game $G(a_1, a_2, \dots, a_m)$ if m is odd and if $a_1 \geq \lceil \frac{n}{2} \rceil$ then the first player wins. This result also has implications regarding the construction in $2^{[n]}$ of cutsets of width $n-1$ using $n-1$ long chains. It follows that, for $n \geq 5$, to construct such a cutset, the initial $\lceil \frac{n}{2} \rceil - 1$ long chains can be chosen arbitrarily. However, it is possible to choose $\lceil \frac{n}{2} \rceil$ long chains in a way that the collection *cannot* be enlarged to a cutset of width $n - 1$.

The proof

Some notation: As already mentioned $[n] = \{1, 2, \dots, n\}$, and $2^{[n]}$ is the set of all subsets of $[n]$. $\binom{[n]}{k}$ will denote the subsets of size k of $[n]$, and for $A \in 2^{[n]}$ the complement of A in $[n]$, that is $[n] - A$, will be denoted by A^c. Given two subsets A, B of $[n]$ with $A \subseteq B$, $[A, B]$ will denote the collection of subsets that contain A and are contained in B. Note that if $|A| = a$ and $|B| = b$ then $[A, B]$ is isomorphic, as a poset, to $2^{[b-a]}$ and hence by Theorem 2 if the number of existing chains in $[A, B]$ is at most $b-a-2$ then there exists a long chain in $[A, B]$ disjoint from all the given ones.

The proof of theorem 3 consists of two parts. We have to show that player I can win the game $G(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor - 1, 1)$, and player two can win the game $G(\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor, 1)$. We use Theorem 2 extensively. For $5 \leq n \leq 7$, (easier) variations of the arguments below will prove the result. Here, we give a complete proof for $n \geq 8$.

Player I wins $G(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor - 1, 1)$. Theorem 2 implies that the second player will always be able to complete her move, and thus the first player will have to construct the first $\lceil \frac{n}{2} \rceil$ chains in a way that does not allow the second player to enlarge the collection to a cutset of width $n - 1$. Let $L = \{\lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \dots, n\} \subset [n]$. L is a subset of size $\lfloor \frac{n}{2} \rfloor$, and 2^L, the set of all subsets of L ordered by inclusion, is isomorphic to $2^{[\lfloor \frac{n}{2} \rfloor]}$. Now let $\mathcal{Q}_L$ be the subsets of $[n]$ that contain L. $\mathcal{Q}_L$ is the same as $[L, [n]]$ and hence is isomorphic to $2^{[\lceil \frac{n}{2} \rceil]}$. Player I will construct $\lfloor \frac{n}{2} \rfloor$ long chains in $2^{[n]}$ as follows: First construct $\lfloor \frac{n}{2} \rfloor$ long chains in 2^L. These chains begin with sets of size one and end with sets of size $\lfloor \frac{n}{2} \rfloor - 1$. Call these latter sets A_1, A_2, …, $A_{\lfloor \frac{n}{2} \rfloor}$. For $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$ extend the ith chain by

$$A_i \subset A_i \cup \{i\} \subset L \cup \{i\}.$$

If n is odd, the first player needs one more chain and this will start as

$$\{\lceil \frac{n}{2} \rceil\} \subset \{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1\} \subset \dots \subset \{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \dots, n\} = L \cup \{\lceil \frac{n}{2} \rceil\}.$$

Player I has now constructed the required number of chains and needs to extend them so that they become long chains. All the chains have stopped at a set of size $|L| + 1$ in $\mathcal{Q}_L$. In fact, every set of size $|L| + 1$ in $\mathcal{Q}_L$ is in one of the chains. Since $\mathcal{Q}_L$ is isomorphic to $2^{[\lceil \frac{n}{2} \rceil]}$ we can, entirely within

$\mathcal{Q}_L$, extend these chains to sets of size $n-1$. Thus player I has completed the construction of $\lceil \frac{n}{2} \rceil$ long chains.

Player II will now construct an additional $\lfloor \frac{n}{2} \rfloor - 1$ chains. One set of size one say $\{v\}$ and one set of size $n-1$ say $\{w\}^c = [n] - \{w\}$ remain unused (i.e., not in any of the chains constructed by I or II). Player I will now have to construct one last long chain starting with $\{v\}$ and ending at $\{w\}^c$. Now, $[\{v\}, \{w\}^c]$, the set of subsets of $[n]$ that contain v but do not contain w, is isomorphic to $2^{[n-2]}$ and contains none of the subsets in $2^{[L]}$ or $\mathcal{Q}_L$. Thus from the original long chains constructed by player I at most one chain (in fact at most one subset in one of the chains) can be in $[\{v\}, \{w\}^c]$. Thus the total number of chains already in $[\{v\}, \{w\}^c]$ is at most $1 + (\lfloor \frac{n}{2} \rfloor - 1) = \lfloor \frac{n}{2} \rfloor$. This number is less than $n-3$ for $n \geq 7$ and thus by theorem 2 there will exist a long chain in $[\{v\}, \{w\}^c]$ disjoint from the previous ones. Player I will win by constructing this chain.

Player II will win the game $G(\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor, 1)$. Player I has chosen $\lceil \frac{n}{2} \rceil - 1$ long chains. Let $\mathcal{B}_I$ and $\mathcal{E}_I$ respectively denote the set of subsets of size one and the set of subsets of size $n-1$ in these chains. Thus we have $|\mathcal{B}_I| = |\mathcal{E}_I| = \lceil \frac{n}{2} \rceil - 1$. Since less than half of the sets of size one in $2^{[n]}$ belong to $\mathcal{B}_I$, there exists $x \in [n]$ such that $\{x\} \notin \mathcal{B}_I$ and $\{x\}^c \notin \mathcal{E}_I$. We will demonstrate that player II can choose her $\lfloor \frac{n}{2} \rfloor$ long chains appropriately so as to leave $\{x\}$ and $\{x\}^c$ as the only unused subsets of size one and size of $n-1$ respectively. There is no chain going from $\{x\}$ to $\{x\}^c$ and hence Player I loses.

Let $\mathcal{B}_{II} = \binom{[n]}{1} - \mathcal{B}_I - \{x\}$ and $\mathcal{E}_{II} = \binom{[n]}{n-1} - \mathcal{B}_{II} - \{x\}^c$. We have $|\mathcal{B}_{II}| = |\mathcal{E}_{II}| = \lfloor \frac{n}{2} \rfloor \geq 4$ and player II's chains will start with an element of $\mathcal{B}_{II}$ and end with an element of $\mathcal{E}_{II}$. Choose two distinct sets $\{a\}, \{l\} \in \mathcal{B}_{II}$, such that for some $e \in \{a, l\}^c$ with $\{e\}^c \in \mathcal{E}_{II}$ we have $\{l, e\}^c$ not in any of chains constructed by player I. A simple count shows that this is possible since, for an arbitrary $\{a\} \in \mathcal{B}_{II}$, if for the first choice of l all the sets of the form $\{l, e\}^c$ are taken by player I's chains then for the second choice of l at most one such set can be taken. So to summarize, $\{a\}, \{l\} \in \mathcal{B}_{II}$, $e \neq a, l$, $\{e\}^c \in \mathcal{E}_{II}$, and $\{l, e\}^c$ is not in any of the chains constructed by player I.

Now $[\{a\}, \{l, e\}^c]$ forms a Boolean lattice of order $n-3$ and for $n \geq 8$ we have $\lceil \frac{n}{2} \rceil - 1 \leq n - 5$, and so by Theorem 2 player II can form a chain starting with $\{a\}$ and ending with $\{l, e\}^c$. This chain can then be continued to $\{e\}^c$ and will be the first chain constructed by player II. Next construct a bipartite graph whose vertices are $(\mathcal{B}_{II} - \{a\}) \cup (\mathcal{E}_{II} - \{e\}^c)$ and two subsets are connected by an edge if one is included in the other. Every element of $\mathcal{B}_{II} - \{a\}$ is not contained in at most one of the elements of $\mathcal{E}_{II} - \{e\}^c$ and hence by the Marriage theorem we can find a complete matching. Thus every element Y of $\mathcal{B}_{II} - \{a\}$ is matched with an element $m(Y)$ of $\mathcal{E}_{II} - \{e\}^c$, in such a way that $Y \subset m(Y)$ and hence $m(Y) \neq Y^c$. Now let $m(\{l\}) = \{d\}^c$ for some $d \in [n]$. Note that $d \neq e$ and d may or may not be in $\mathcal{B}_{II}$. Finally choose $\{b\} \in \mathcal{B}_{II} - \{\{a\}, \{l\}, \{d\}\}$. Now for each $\{z\} \in \mathcal{B}_{II} - \{\{a\}, \{b\}, \{l\}\}$, we have that $[\{z\}, m(z)]$ is isomorphic to $2^{[n-2]}$ and at most (the largest number being for the final $\{z\}$ in this collection) there are already $(\lceil \frac{n}{2} \rceil - 1) + 1 + (\lfloor \frac{n}{2} \rfloor - 4) = n - 4$ chains in this Boolean lattice. By Theorem 2 we can find a chain from each $\{z\} \in \mathcal{B}_{II} - \{\{a\}, \{l\}, \{d\}\}$ to $m(z)$. Player II will construct all of these. At this point $\{b\}, \{l\}, \{x\}$ are the unused subsets of size one and $m(b), m(l) = \{d\}^c$, and $\{x\}^c$ are the unused subsets of size $n-1$. Note that by construction $\{b\}^c$ is not one of the unused subsets and hence there must be a chain that ends at this subset. This chain cannot be in the Boolean lattice $[\{b\}, m(b)]$ and hence at most $(n-3)-1$ chains are in this Boolean lattice of order $n-2$. Again, by Theorem 2, player II will be able to find a chain from $\{b\}$ to $m(b)$. It remains to show that the number of constructed chains in $[\{l\}, \{d\}^c]$ is at most $n-4$ so as to allow a final application of Theorem 2 for the construction of player II's final chain.

The total number of chains at this point is $n-2$ and we claim that two of these chains cannot be in $[\{l\},\{d\}^c]$ completing the proof. First of all the subset $\{l\}^c$ is not open and hence one of the given chains must end at this subset and thus cannot be in $[\{l\},\{d\}^c]$. In addition, player II's first chain that ended in $\{e\}^c$ had gone from $\{a\}$ to $\{l,e\}^c$ and hence could not be in $[\{l\},\{d\}^c]$. So two of the $n-2$ existing chains are not in this Boolean lattice and thus player II can find a chain from $\{l\}$ to $\{d\}^c$. This leaves $\{x\}$ and $\{x\}^c$ open for player I, and so player II wins as claimed. □

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