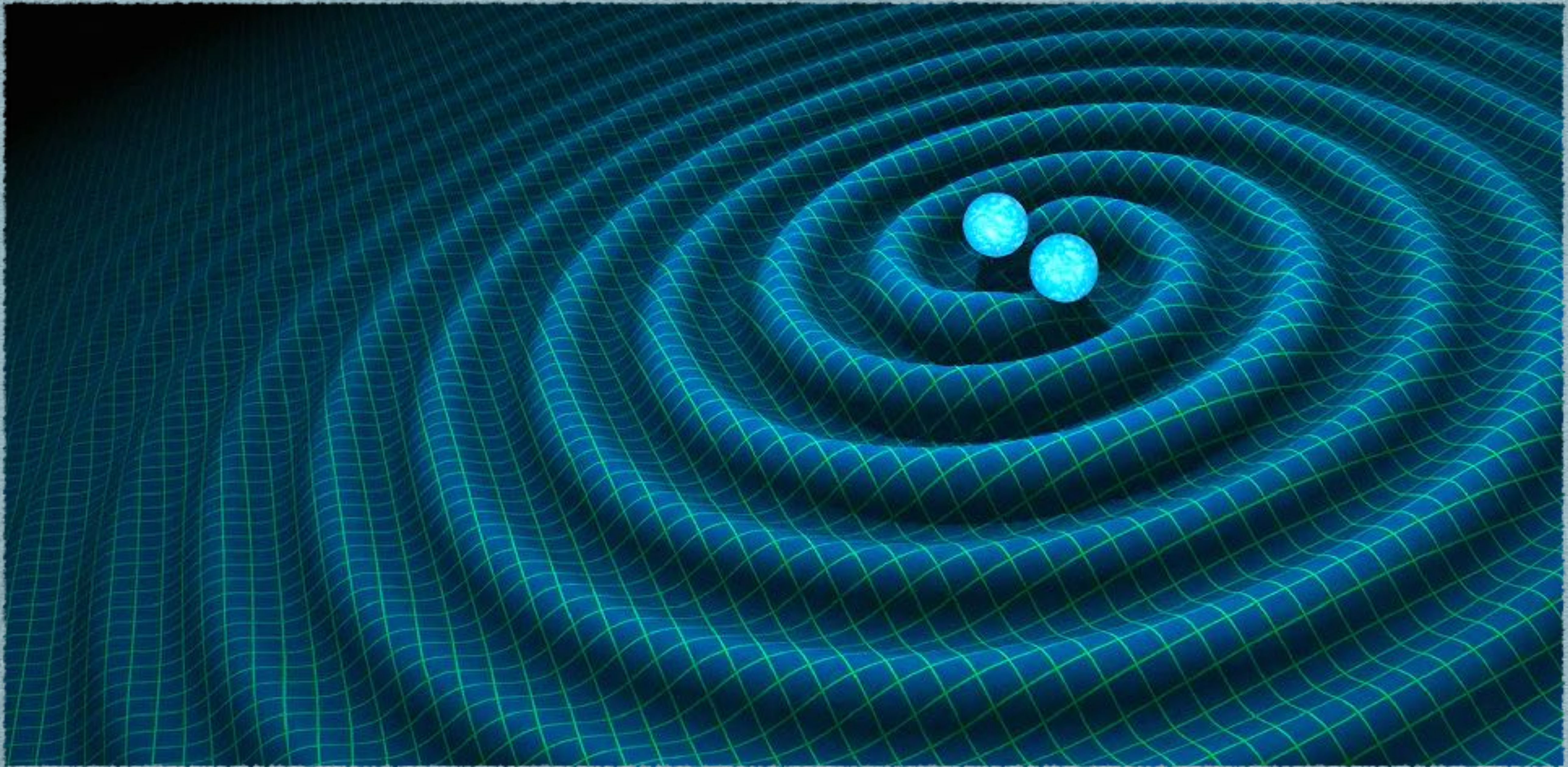


General Relativity and Gravitational Waves:

Session 1. Overview and Special Relativity

Thomas A. Moore — Les Houches — July 3, 2018

Overview of the Series

1. Overview and Special Relativity
2. General Coordinates
3. The Einstein Equation
4. Solving the Einstein Equation
5. Gravitational Waves

Overview of this session:

1.3 Overview of General Relativity

1.4 The Geometric Analogy and the Metric Equation

1.5 Four-Vectors and Summation Notation

1.6 Tensors and Covariant Equations

1.7 Maxwell's Equations

Simple principles:

Special Relativity:

1. The Principle of Relativity
2. *c* is Frame-Independent

General Relativity:

3. The Geodesic Hypothesis
4. The Principle of Coordinate Independence

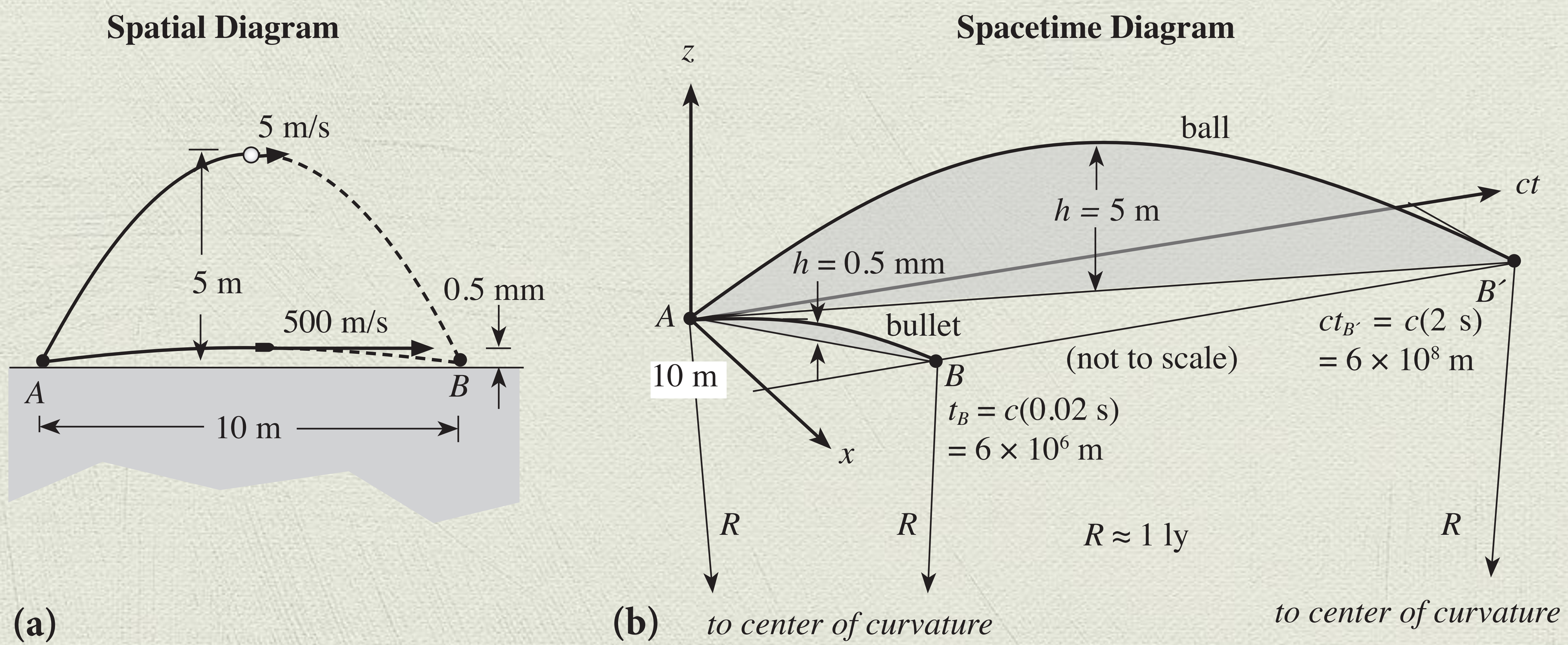
The geodesic principle:

"A free object follows a geodesic in spacetime."

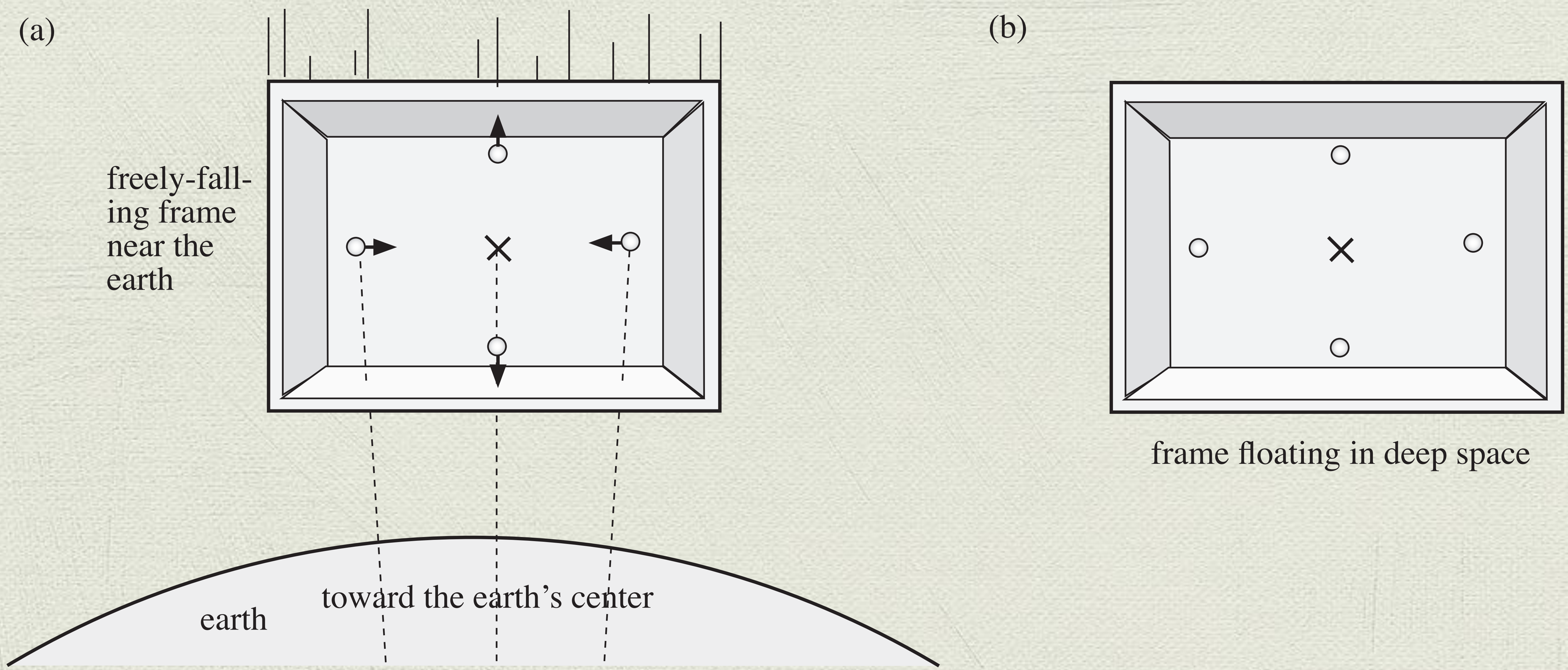
Notes:

1. A geodesic is the "straightest possible curve"
2. A "free" object is one that does not interact with anything else.
3. This only works in *spacetime*.

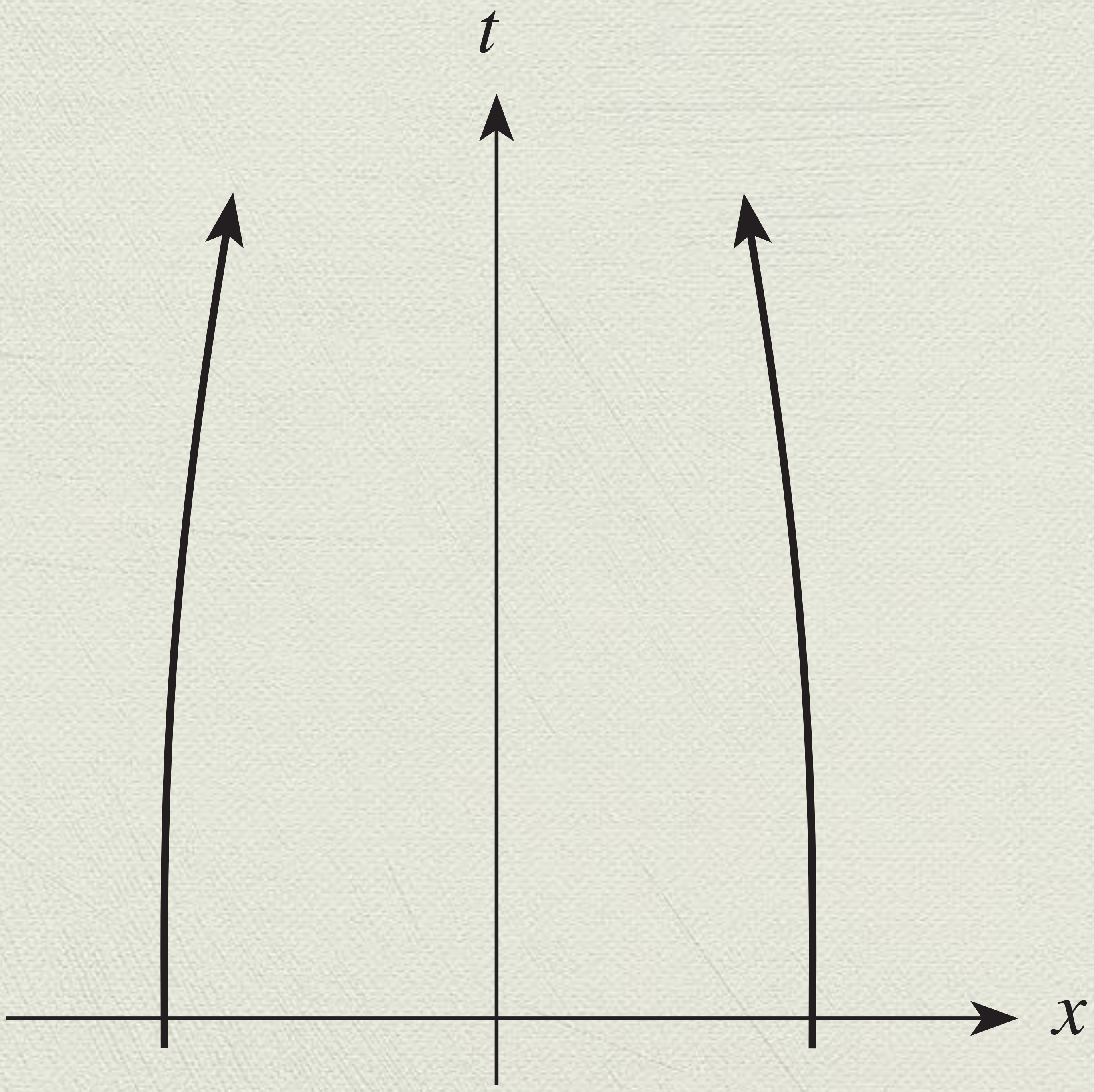
Paths in Space and Spacetime:



What is frame-independent?



Spacetime is Curved:



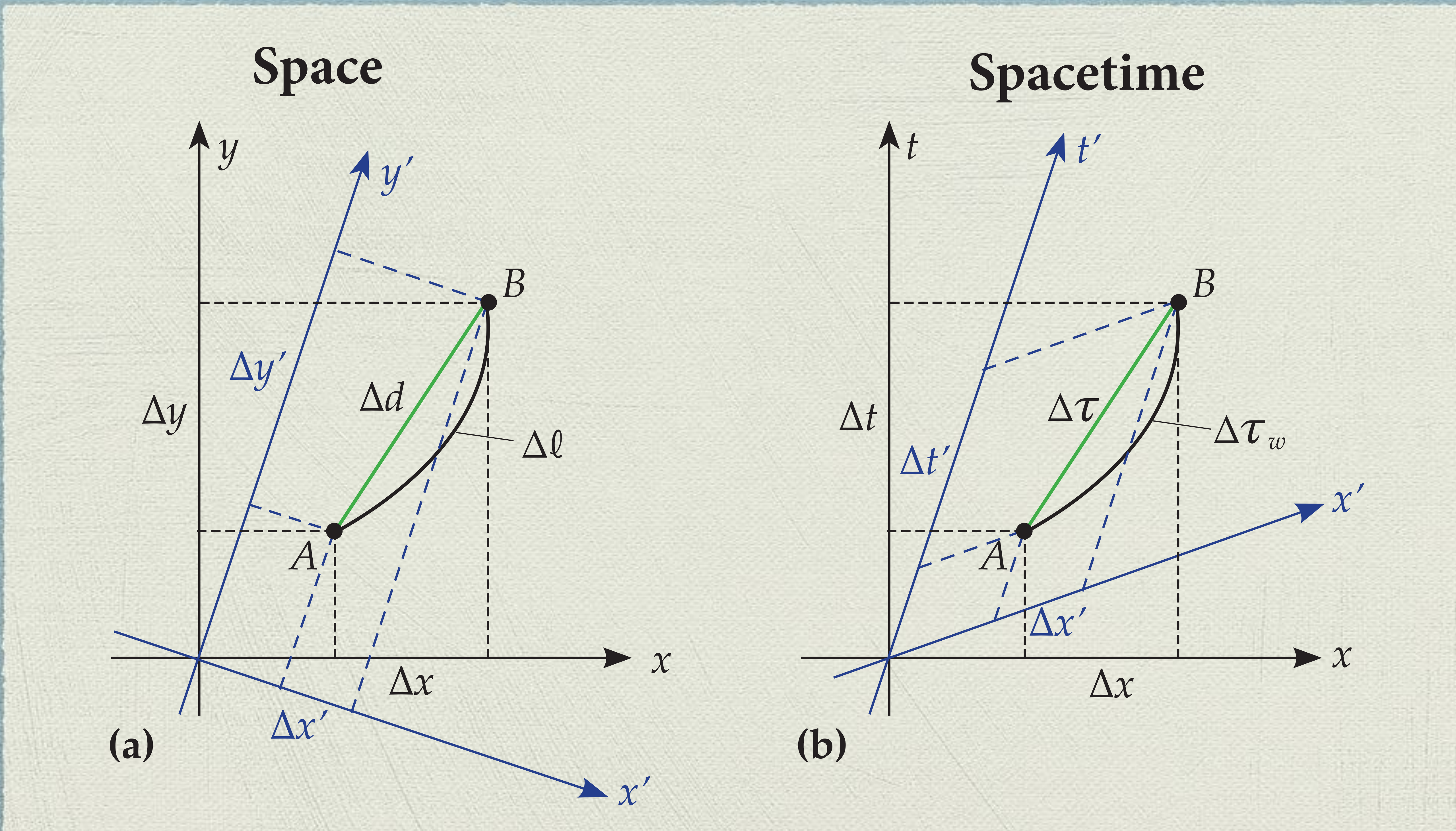
The Two Core Equations:

Einstein Equation: $$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

Geodesic Equation: $$\frac{d^2 x^\alpha}{d\tau^2} = \Gamma^\alpha_{\mu\nu} u^\mu u^\nu$$

"Spacetime tells matter how to move;
Matter tells spacetime how to curve."

The Geometric Analogy



Frame-independent separations:

In space: **the Pythagorean Equation**

$$\Delta d^2 = \Delta x^2 + \Delta y^2 = (\Delta x')^2 + (\Delta y')^2$$

In spacetime: **the Metric Equation**

$$\begin{aligned}\Delta\tau^2 &= \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \\ &= (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2\end{aligned}$$

Proper time along a worldline:

$$\Delta\tau_w = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \int \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int \sqrt{1 - v^2}\, dt \qquad (1.5)$$

Types of intervals:

$$\begin{array}{rl} \textbf{Timelike:} & \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 > 0 \\ \textbf{Spacelike:} & \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 < 0 \\ \textbf{Timelike:} & \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0 \end{array} \tag{1.6}$$

Spacetime separation:

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \quad (= -\Delta \tau^2) \tag{1.7}$$

$$\begin{array}{rl} \textbf{Timelike:} & \Delta s^2 < 0 \\ \textbf{Spacelike:} & \Delta s^2 > 0 \\ \textbf{Timelike:} & \Delta s^2 = 0 \end{array} \tag{1.8}$$

Exercise!

1.4.1 Exercise: The Three Kinds of Time.

Alice drives a race car around a track. Bob stands at a fixed position beside the track. Let event A be Alice passing Bob the first time and event B be Alice passing Bob the next time. Both Alice and Bob measure the time between these events with their watches. Now, Cara and David are riding a train whose track passes very close to Bob's position and which is moving at a constant velocity. It happens that Cara passes Bob just as event A occurs and David passes Bob just as event B occurs. Cara and David note the times of these events on their watches, which have been previously synchronized in the train frame. They determine the time between the events by calculating the difference in the times they measure. (Assume that the ground frame is adequately inertial for events occurring in a plane perpendicular to the earth's gravitational field.)

(a) Who measures a coordinate time between these events in some inertial reference frame?
(b) Who measures a proper time between these events along a worldline that connects the events?
(c) Who measures the spacetime interval between the events?
(d) Who measures the shortest time interval between these events?
(e) Who measures the longest time interval between these events?

Choices are: A. Alice B. Bob C. Cara and David (A question may have multiple answers.)

Lorentz Transformation

$$\begin{aligned} t' &= \gamma t - \gamma\beta x \qquad & t &= \gamma t' + \gamma\beta x' \\ x' &= -\gamma\beta t + \gamma x \qquad & x &= \gamma\beta t' + \gamma x' \\ y' &= y \qquad & y &= y' \\ z' &= z \qquad & z &= z' \end{aligned} \tag{1.9}$$

$$\Lambda^\mu{}_\nu = \begin{array}{cc} & \begin{array}{cccc} \nu=t & x & y & z \end{array} \\ \begin{array}{c} \mu=t \\ x \\ y \\ z \end{array} & \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (\Lambda^{-1})^\mu{}_\nu = \begin{array}{cc} & \begin{array}{cccc} \nu=t & x & y & z \end{array} \\ \begin{array}{c} \mu=t \\ x \\ y \\ z \end{array} & \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{1.10}$$

$$x'^\mu = \sum_{\nu=t,x,y,z} \Lambda^\mu{}_\nu\, x^\nu \qquad \text{and} \qquad x^\mu = \sum_{\nu=t,x,y,z} (\Lambda^{-1})^\mu{}_\nu\, x'^\nu \tag{1.11}$$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \qquad \text{and} \qquad x^\mu = (\Lambda^{-1})^\mu{}_\nu\, x'^\nu \tag{1.12}$$

$$\Delta x'^\mu = \Lambda^\mu{}_\nu\, \Delta x^\nu \qquad \text{and} \qquad \Delta x^\mu = (\Lambda^{-1})^\mu{}_\nu\, \Delta x'^\nu \tag{1.13}$$

Lorentz Transformation

$$\begin{bmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \tag{1.14}$$

$$\Delta x^\mu = (\Lambda^{-1})^\mu{}_\nu (\Lambda^\nu{}_\alpha \, \Delta x^\alpha) \quad \text{which in turn implies that} \quad (\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\alpha = \delta^\mu{}_\alpha \tag{1.15}$$

Metric Tensor and equation:

$$\eta_{\mu\nu} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.16}$$

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{1.17}$$

Four-Vectors

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu}\, A^{\nu} \tag{1.18}$$

$$A^2 \equiv \eta_{\mu\nu} A^{\mu} A^{\nu} \equiv \boldsymbol{A \cdot A} \tag{1.19}$$

$$u^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau} \tag{1.20}$$

$$\boldsymbol{u \cdot u} = \eta_{\mu\nu} u^{\mu} u^{\nu} = \eta_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\mu}}{d\tau} = \frac{\eta_{\mu\nu} dx^{\mu} dx^{\nu}}{d\tau^2} = \frac{-d\tau^2}{d\tau^2} = -1 \tag{1.21}$$

$$u^t = \frac{dt}{d\tau} = \frac{dt}{dt\sqrt{1-v^2}} = \frac{1}{\sqrt{1-v^2}}, \quad u^x = \frac{dx}{d\tau} = \frac{dx}{dt\sqrt{1-v^2}} = \frac{v_x}{\sqrt{1-v^2}} \tag{1.23}$$

$$\boldsymbol{p} = m\boldsymbol{u} \quad \Rightarrow \quad p^{\alpha} = mu^{\alpha} = m\frac{dx^{\alpha}}{d\tau} \quad \Rightarrow \quad \boldsymbol{p \cdot p} = m^2(\boldsymbol{u \cdot u}) = m^2(-1) = -m^2 \tag{1.24}$$

Gradient as a Covector

$$\frac{\partial \Phi}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial \Phi}{\partial t} + \frac{\partial t}{\partial x'}\frac{\partial \Phi}{\partial x} + \frac{\partial t}{\partial y'}\frac{\partial \Phi}{\partial y} + \frac{\partial t}{\partial z'}\frac{\partial \Phi}{\partial z}$$

$$= (\Lambda^{-1})^t{}_t \frac{\partial \Phi}{\partial t} + (\Lambda^{-1})^x{}_t \frac{\partial \Phi}{\partial x} + (\Lambda^{-1})^y{}_t \frac{\partial \Phi}{\partial y} + (\Lambda^{-1})^z{}_t \frac{\partial \Phi}{\partial z} \quad (1.25)$$

$$\partial'_\alpha \Phi = (\Lambda^{-1})^\beta{}_\alpha \, (\partial_\beta \Phi) \quad (1.26)$$

Tensors

Gradient of a Four-Vector:

$$\partial'_\alpha A'^{\,\beta} = (\Lambda^{-1})^\mu{}_\alpha \frac{\partial}{\partial x^\mu}\left(\Lambda^\beta{}_\nu A^\nu\right) = (\Lambda^{-1})^\mu{}_\alpha \Lambda^\beta{}_\nu \left(\partial_\mu A^\nu\right) \tag{1.27}$$

General Tensor Transformation Rule:

$$T'^{\alpha\cdots}{}_{\beta\cdots}{}^{\gamma\cdots} = \Lambda^\alpha{}_\mu \cdots (\Lambda^{-1})^\nu{}_\beta \cdots \Lambda^\gamma{}_\sigma \cdots T^{\mu\cdots}{}_{\nu\cdots}{}^{\sigma\cdots} \tag{1.28}$$

Tensor Operations

Tensor Product:

$$T'^{\,\mu\nu} \equiv A'^{\,\mu} B'^{\,\nu} = (\Lambda^{\mu}{}_{\alpha} A^{\alpha})(\Lambda^{\nu}{}_{\beta} B^{\beta})$$

$$= \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} (A^{\alpha} B^{\beta}) = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta} \qquad (1.29)$$

Contraction:

$$T'^{\alpha}{}_{\alpha} = \Lambda^{\alpha}{}_{\mu} (\Lambda^{-1})^{\nu}{}_{\alpha} T^{\mu}{}_{\nu} = \delta^{\nu}{}_{\mu} T^{\mu}{}_{\nu} = T^{\nu}{}_{\nu} \quad (1.30,\ 31)$$

(These yield nothing useful: $\sum T_{\mu\mu}$ and $\sum T^{\mu\mu}$)

The Metric is a tensor

$$\eta_{\mu\nu}\Delta x'^{\,\mu}\Delta x'^{\,\nu} = \eta_{\alpha\beta}\Delta x^{\alpha}\Delta x^{\beta} = \eta_{\alpha\beta}(\Lambda^{-1})^{\alpha}{}_{\gamma}\,\Delta x'^{\,\gamma}(\Lambda^{-1})^{\beta}{}_{\sigma}\,\Delta x'^{\,\sigma}$$

$$= [(\Lambda^{-1})^{\alpha}{}_{\gamma}(\Lambda^{-1})^{\beta}{}_{\sigma}\eta_{\alpha\beta}]\Delta x'^{\,\gamma}\Delta x'^{\,\sigma} \qquad (1.33)$$

$$\eta_{\mu\nu}\Delta x'^{\,\mu}\Delta x'^{\,\nu} = [(\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}\eta_{\alpha\beta}]\Delta x'^{\,\mu}\Delta x'^{\,\nu}$$

$$0 = [\eta_{\mu\nu} - (\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}\eta_{\alpha\beta}]\Delta x'^{\,\mu}\Delta x'^{\,\nu} \qquad (1.34)$$

$$0 = \eta_{\mu\nu} - (\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}\eta_{\alpha\beta}$$

$$\Rightarrow \quad \eta_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}\eta_{\alpha\beta}$$

Tensor Operations

Inverse Metric: $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}{}_{\nu}$

Contraction: $A_\mu = \eta_{\mu\nu}A^\nu$ and $B^\mu = \eta^{\mu\nu}B_\nu$

Addition:
$$\begin{aligned} C'^{\,\mu} &= A'^{\,\mu} + B'^{\,\mu} = \Lambda^{\mu}{}_{\nu}A^\nu + \Lambda^{\mu}{}_{\alpha}B^\alpha \\ &= \Lambda^{\mu}{}_{\nu}(A^\nu + B^\nu) = \Lambda^{\mu}{}_{\nu}C^\nu \end{aligned}$$

Tensor Addition: example: $p^\mu_{\text{tot}} = p^\mu_1 + p^\mu_2$
Tensor Product: example: $A^\mu B^\nu = T^{\mu\nu}$
Contraction: example: $\delta^{\mu}{}_{\mu} = 4$
Lowering indices: example: $A_\mu = \eta_{\mu\nu}A^\nu$
Raising indices: example: $B^\mu = \eta^{\mu\nu}B_\nu$
Renaming summed indices: example: $\delta^{\mu}{}_{\mu} = \delta^{\nu}{}_{\nu} = 4$

Why tensors?

A tensor equation is **absolute**:
it has the same form in every inertial reference frame.

Example:

$$\frac{d}{d\tau}(\eta_{\mu\nu} p^\mu p^\nu) = \frac{d}{d\tau}(p_\mu p^\mu) = 0$$

expresses the physical fact that the magnitude of a particle's four-momentum (its mass) is conserved in all frames.

Some terminology:

$$A_\mu = \eta_{\mu\nu} A^\nu$$

Free index

Bound index

A free index tells you how many equations:

$$A_t = \eta_{t\nu} A^\nu = \eta_{tt} A^t + \eta_{tx} A^x + \eta_{tx} A^y + \eta_{tz} A^z$$
$$A_x = \eta_{x\nu} A^\nu = \eta_{xt} A^t + \eta_{xx} A^x + \eta_{xy} A^y + \eta_{xz} A^z$$
$$A_y = \eta_{y\nu} A^\nu = \eta_{yt} A^t + \eta_{yx} A^x + \eta_{yy} A^y + \eta_{yz} A^z$$
$$A_z = \eta_{z\nu} A^\nu = \eta_{zt} A^t + \eta_{zx} A^x + \eta_{zy} A^y + \eta_{zz} A^z$$

Some terminology:

Terms

$$\frac{d}{d\tau}(\eta_{\mu\nu}p^{\mu}p^{\nu}) = \eta_{\mu\nu}\frac{dp^{\mu}}{d\tau}p^{\nu} + \eta_{\mu\nu}p^{\mu}\frac{dp^{\nu}}{d\tau}$$

Factors Factors

The Rules

1. **Free indices:**
   - You can't add or equate items with different numbers and/or positions of free indices.
   - Free index names must agree

Bad: $A^2 = \eta_{\mu\nu} A^{\alpha\beta}$ $\qquad A^\mu = B^\nu$ $\qquad A_\mu = B^\mu$

The only exception:

Zero has as many indices in whatever positions you want

$p^\mu_{\text{tot}} = 0$

The Rules

2. **Renaming free indices:** You may rename a free index with a different index name as long as:
   - The name does not collide with any other free index name or any bound index name
   - You rename every occurrence of that index name

Bad:

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \quad \rightarrow \quad A'^{\alpha} = \Lambda^{\mu}{}_{\nu} A^{\nu}$$

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \quad \rightarrow \quad A'^{\alpha} = \Lambda^{\alpha}{}_{\nu} A^{\nu}$$

$$A_{\mu} = \eta_{\mu\nu} A^{\nu} \quad \rightarrow \quad A_{\nu} = \eta_{\nu\nu} A^{\nu}$$

The Rules

3. **Renaming bound indices:** You may rename a bound index with a different index name as long as:
   - The name does not collide with any free index name or any bound index name *in the same term*
   - You rename both occurrences of that index name

Bad: $A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \quad \rightarrow \quad A'^{\mu} = \Lambda^{\mu}{}_{\mu} A^{\mu}$

Good:
$$\begin{aligned} C'^{\,\mu} &\equiv A'^{\,\mu} + B'^{\,\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} + \Lambda^{\mu}{}_{\alpha} B^{\alpha} \\ &= \Lambda^{\mu}{}_{\nu}(A^{\nu} + B^{\nu}) \equiv \Lambda^{\mu}{}_{\nu} C^{\nu} \end{aligned}$$

4. **When in doubt, write it out.**

Exercise!

1.6.1 Exercise: Good or Bad?

Consider the equations listed below. Answer A = Violates Rule 1, B = Violates Rule 2, C = Violates Rule 3, D = OK for each equation. For each acceptable equation, specify how many equations it implicitly represents (A = 1, B = 4, C = 16, D = 64, E = 256).

(a) $\eta_{\mu\nu} u^\mu u^\nu = -1$

(b) $p^\alpha = m u^\alpha \delta^\mu_\beta$

(c) $\Lambda^\alpha{}_\beta A^\beta = A'^{\,\alpha}$ renamed to $\Lambda^\alpha{}_\beta A^\beta = A'^{\,\mu}$

(d) $\eta_{\mu\nu} (\Lambda^{-1})^\nu{}_\alpha = \eta_{\mu\beta} \Lambda^\beta{}_\alpha$

(e) $\eta_{\mu\nu} A^\mu B^\nu = 0$ renamed to $\eta_{\mu\mu} A^\mu B^\mu = 0$

(f) $\dfrac{dp^\alpha}{d\tau} = q F^{\mu\nu} u^\alpha$

(g) $T_{\mu\nu\alpha} + T_{\alpha\mu\nu} + T_{\nu\alpha\mu} = 0$

(h) $0 = \eta_{\mu\nu} A^\mu B^\nu + \eta_{\alpha\beta} A^\alpha C^\beta$ renamed to $0 = \eta_{\mu\nu} A^\mu B^\nu + \eta_{\mu\nu} A^\mu C^\nu$

Maxwell's Equations: Our starting point

Equations for a electrostatic field:

$$\frac{d\vec{p}}{dt} = -q\vec{\nabla}\phi \qquad\qquad -\nabla^2\phi = \frac{\rho}{\varepsilon_0}$$

We seek tensor generalizations that reduce to these equations in the static limit.

What kind of quantity is the charge density?

$$\rho' = \frac{q}{V'} = \frac{q}{V\sqrt{1-\beta^2}} = \gamma\rho$$

Maxwell's equations: the current density four-vector

Define the four-current density ***J***:

$$J^t \equiv \rho, J^x = \rho v_x, J^y = \rho v_y, J^z = \rho v_z.$$

$$\rho' = J'^{\,t} = \gamma J^t - \gamma\beta J^x = \gamma\rho - 0 = \gamma\rho$$

$$J'^{\,x} = -\gamma\beta J^t + \gamma J^x = -\gamma\beta\rho + 0 =$$

$$J'^{\,y} = J^y = 0$$

$$J'^{\,z} = J^z = 0$$

So this quantity transforms like a four-vector

(Charge density is the time component of this four-vector)

Maxwell's Equations: Generalizing the Poisson equation

Note that: $-\partial_\mu \partial^\mu \equiv -\eta^{\mu\nu} \partial_\mu \partial_\nu = \ +\partial^2/\partial t^2 - \nabla^2$

So Poisson's equation generalizes to:

$$-\partial_\mu \partial^\mu A^\alpha = \frac{1}{\varepsilon_0} J^\alpha$$

or even more generally to

$$-\partial_\mu (\partial^\mu A^\alpha + b\, \partial^\alpha A^\mu) = \frac{1}{\varepsilon_0} J^\alpha$$

because

$$-\partial_\mu (\partial^\mu A^t + b\, \partial^t A^\mu) = \frac{1}{\varepsilon_0} J^t \quad \Rightarrow \quad -\partial_\mu (\partial^\mu \phi + b \cdot 0) = \frac{1}{\varepsilon_0} \rho$$

$$\Rightarrow \quad -\nabla^2 \phi = \frac{1}{\varepsilon_0} \rho$$

Maxwell's Equations: Generalizing the force equation

First try: $\frac{dp^\alpha}{d\tau} = -q\,\partial^\alpha A^\mu$

Note that for a particle at rest:

$$u^t = 1, u^x = u^y = u^z = 0 \quad \Rightarrow \quad u_t = \eta_{t\nu} u^\nu = \eta_{tt} u^t + 0 = -1$$

So a better guess is:

$$\frac{dp^\alpha}{d\tau} = +q\,\partial^\alpha A^\mu u_\mu$$

More general is:

$$\frac{dp^\alpha}{d\tau} = q(\partial^\alpha A^\mu + h\,\partial^\mu A^\alpha) u_\mu$$

Maxwell's Equations: Constraining h

Note that:

$$0 = \frac{d}{d\tau}(-m^2) = \frac{d}{d\tau}(p^\alpha \eta_{\alpha\beta} p^\beta) = \frac{dp^\alpha}{d\tau}\eta_{\alpha\beta}p^\beta + p^\alpha \eta_{\alpha\beta}\frac{dp^\beta}{d\tau}$$

$$= \frac{dp^\alpha}{d\tau}\eta_{\alpha\beta}p^\beta + p^\beta \eta_{\beta\alpha}\frac{dp^\alpha}{d\tau} = 2\frac{dp^\alpha}{d\tau}\eta_{\alpha\beta}p^\beta$$

So if we multiply both sides of

$$\frac{dp^\alpha}{d\tau} = q(\partial^\alpha A^\mu + h\,\partial^\mu A^\alpha)u_\mu$$

by $\eta_{\alpha\beta}p^\beta = p_\alpha = mu_\alpha$ we get

$$0 = 2qm(\partial^\alpha A^\mu + h\,\partial^\mu A^\alpha)u_\alpha u_\mu$$

so $h = -1$.

Maxwell's Equations: Constraining b

Similarly,

$$\frac{1}{\varepsilon_0}\partial_\alpha J^\alpha = -\partial_\alpha\partial_\mu(\partial^\mu A^\alpha + b\,\partial^\alpha A^\mu) = -\partial_\alpha\partial_\mu\partial^\mu A^\alpha - b\,\partial_\mu\partial_\alpha\partial^\mu A^\alpha$$
$$= -(1+b)\partial_\alpha\partial_\mu\partial^\mu A^\alpha$$

But charge conservation requires that

$$\partial_\alpha J^\alpha = \partial\rho/\partial t + \partial J^x/\partial x + \partial J^y/\partial y + \partial J^z/\partial z = 0$$

so $b = -1$.

Maxwell's Equations: Final equations for the potential

So we have

$$\frac{dp^\alpha}{d\tau} = q(\partial^\alpha A^\mu - \partial^\mu A^\alpha)u_\mu \qquad \partial_\mu(\partial^\alpha A^\mu - \partial^\mu A^\alpha) = \frac{1}{\varepsilon_0} J^\alpha$$

Define $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\equiv \begin{array}{c c} & \begin{array}{cccc} \nu{=}t & x & y & z \end{array} \\ \begin{array}{c} \mu{=}t \\ x \\ y \\ z \end{array} & \left[\begin{array}{cccc} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_z & -B_x & 0 \end{array}\right] \end{array}$$

consistent with

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Maxwell's Equations: Final equations for the fields

So we have

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\mu}u_\mu \quad \text{and} \quad \partial_\mu F^{\alpha\mu} = \frac{1}{\varepsilon_0}J^\alpha$$

The first is the Lorentz force law:

$$\begin{aligned}\frac{dp^x}{d\tau} &= q(F^{xt}u_t + F^{xx}u_x = F^{xy}u_y + F^{xz}u_z) \\ &= q\frac{1}{\sqrt{1-v^2}}(-E_x(-1) + 0 + B_z v_y - B_y v_z) \\ &= q(-E_x(-1) + 0 + B_z v_y - B_y v_z) = qE_x + q(\vec{v}\times\vec{B})_x\end{aligned}$$

Maxwell's Equations: Final equations for the fields

So we have

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\mu}u_\mu \quad \text{and} \quad \partial_\mu F^{\alpha\mu} = \frac{1}{\varepsilon_0}J^\alpha$$

The time component of the second is Gauss's Law:

$$\partial_t F^{tt} + \partial_x F^{tx} + \partial_y F^{ty} + \partial_z F^{tz} = 0 + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y} = \frac{J^t}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}$$

The other components are the Ampere-Maxwell law.
The definitions of the field in terms of the potentials imply:

$$\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} = 0$$

which yield the other two Maxwell equations.

Maxwell's Equations: Summary:

We have "derived" Maxwell's Equations by finding tensor generalizations of Newtonian equations for static fields that are:

1. automatically consistent with the POR
2. consistent with the idea that charge is a relativistic scalar
3. consistent with charge conservation
4. consistent with the idea that EM fields don't affect a particle's mass

Exercise!

1.7.1 Exercise: Gauss's law for the Magnetic Field.

Find one choice of values for the indices α, μ, and ν in equation 1.66 that yields Gauss's law for the magnetic field. Are there other choices that yield the same? How many copies of this equation do you think we have in equation 1.66?

$$\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} = 0$$

$$F^{\mu\nu} \equiv \begin{array}{c c} & \begin{array}{cccc} \nu{=}t & x & y & z \end{array} \\ \begin{array}{c} \mu{=}t \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_z & -B_x & 0 \end{bmatrix} \end{array}$$