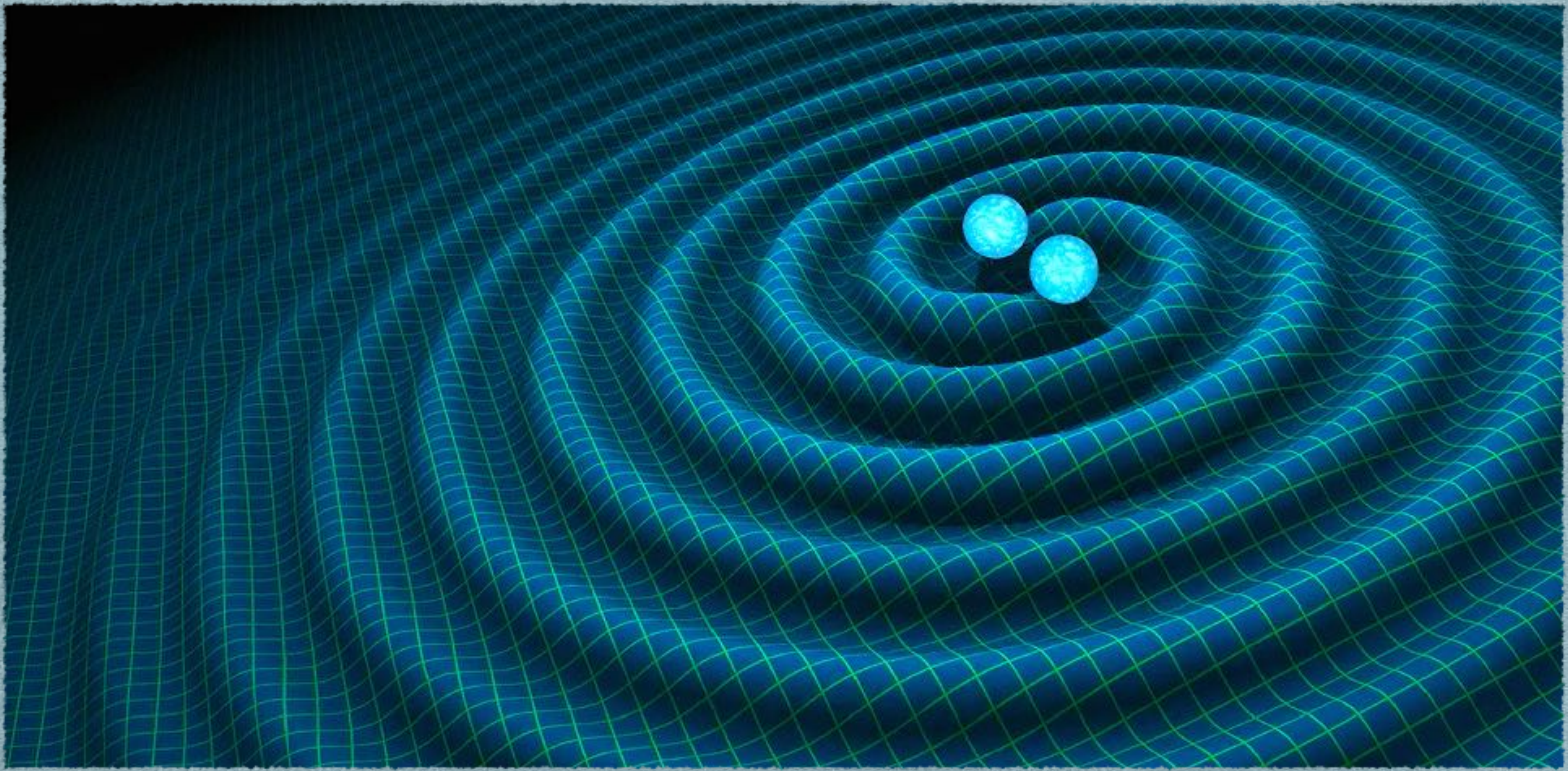


General Relativity and Gravitational Waves:

Session 2. General Coordinates

Thomas A. Moore — Les Houches — July 4, 2018

Overview of this session:

2.2 Definition of a Coordinate Basis

2.3 Tensors in a Coordinate Basis

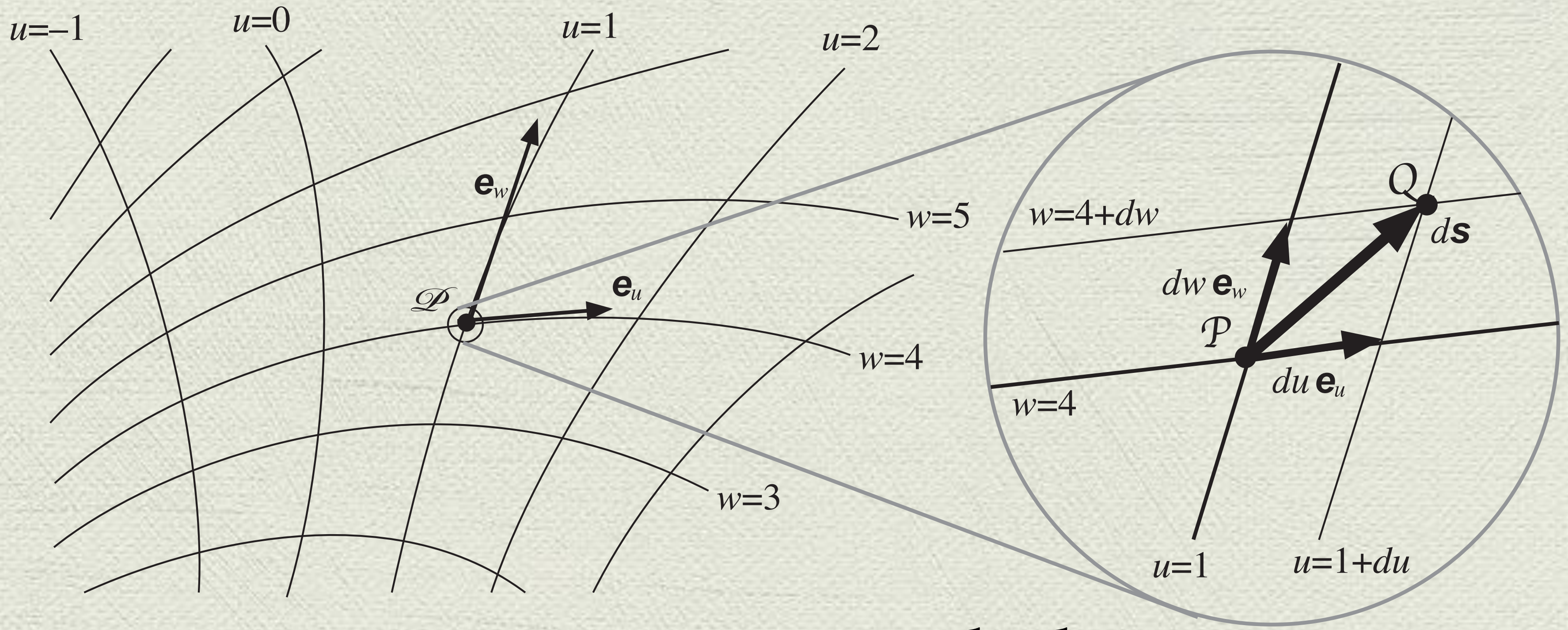
2.4 The Tensor Gradient

2.5 The Geodesic Equation

2.6 Schwarzschild Geodesics

2.7 LOFs and LIFs

Coordinate Basis



Define basis vectors $\mathbf{e}_u$, $\mathbf{e}_w$ such that

1. $\mathbf{e}_u$ points tangent to the w = constant curve in the direction of increasing u
2. $\mathbf{e}_w$ points tangent to the u = constant curve in the direction of increasing w
3. $d\mathbf{s} = du\,\mathbf{e}_u + dw\,\mathbf{e}_w = dx^\mu \mathbf{e}_\mu$

Coordinate Basis

Define a four-vector's components so that

$$\boldsymbol{A} = A^u \boldsymbol{e}_u + A^w \boldsymbol{e}_w = A^\mu \boldsymbol{e}_\mu$$

The metric in a coordinate basis is

$$\begin{aligned} ds^2 &= d\boldsymbol{s} \cdot d\boldsymbol{s} = (du\, \boldsymbol{e}_u + dw\, \boldsymbol{e}_w) \cdot (du\, \boldsymbol{e}_u + dw\, \boldsymbol{e}_w) \\ &= du^2 \boldsymbol{e}_u \cdot \boldsymbol{e}_u + du\, dw\, \boldsymbol{e}_u \cdot \boldsymbol{e}_w + dw\, dw\, \boldsymbol{e}_w \cdot \boldsymbol{e}_u + dw^2\, \boldsymbol{e}_w \cdot \boldsymbol{e}_w \\ &= dx^\mu dx^\nu\, \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \end{aligned}$$

$$g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu$$

Exercise

2.2.1 Exercise: The Metric for Spherical Coordinates

Consider θ-ϕ coordinates on the surface of a sphere of radius R, where curves of constant θ and ϕ are lines of latitude and longitude, respectively (but assume that $\theta = 0$ at the north pole, as is normal in physics, rather than at the equator). Note that these curves are perpendicular everywhere but the poles. By considering what the formula for ds^2 between infinitesimally-separated points must be in terms of $d\theta$ and $d\phi$, find the metric components $g_{\theta\theta}, g_{\theta\phi}, g_{\phi\theta}$, and $g_{\phi\phi}$ as a function of position for a coordinate basis based on the coordinates θ and ϕ. Also, what are the lengths of the $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ basis vectors as a function of position?

Tensors in a coordinate basis

Transformation from u, w to p, q coordinates

$$dp = \frac{\partial p}{\partial u} du + \frac{\partial p}{\partial w} dw \quad \text{and} \quad dq = \frac{\partial q}{\partial u} du + \frac{\partial q}{\partial w} dw$$

or abstractly:

$$dx'^{\,\mu} = \frac{\partial x'^{\,\mu}}{\partial x^\nu} dx^\nu$$

Vector transformation law:

$$A'^{\,\mu} = \frac{\partial x'^{\,\mu}}{\partial x^\nu} A^\nu$$

Generalizes Lorentz transformation, because for a LT:

$$\frac{\partial x'^{\,\mu}}{\partial x^\nu} = \Lambda^\mu{}_\nu$$

Tensors in a coordinate basis

An important identity:

$$\frac{\partial x'^{\,\mu}}{\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\,\alpha}} = \delta^{\mu}{}_{\alpha}$$

Example:

$$\frac{\partial p}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial p} = \frac{dp}{dp} = 1, \qquad \frac{\partial p}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial q} = \frac{dp}{dq} = 0,$$

$$\frac{\partial q}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial q}{\partial w}\frac{\partial w}{\partial p} = \frac{dq}{dp} = 0, \qquad \frac{\partial q}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial q}{\partial w}\frac{\partial w}{\partial q} = \frac{dq}{dq} = 1$$

General tensor transformation law:

$$T'^{\alpha\cdots}{}_{\beta\cdots}{}^{\gamma\cdots} = \frac{\partial x'^{\,\alpha}}{\partial x^{\mu}}\cdots\frac{\partial x^{\nu}}{\partial x'^{\,\beta}}\cdots\frac{\partial x'^{\,\gamma}}{\partial x^{\sigma}}\cdots T^{\mu\cdots}{}_{\nu\cdots}{}^{\sigma\cdots}$$

Tensors in a coordinate basis

Example: the metric tensor is a tensor

$$
\begin{aligned}
g'_{\mu\nu} dx'^{\,\mu} dx'^{\,\nu} &= g_{\alpha\beta} dx^\alpha dx^\beta \\
&= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial x'^{\,\gamma}} dx'^{\,\alpha} \right) \left(\frac{\partial x^\beta}{\partial x'^{\,\sigma}} dx^\sigma \right) \\
&= \left(\frac{\partial x^\alpha}{\partial x'^{\,\gamma}} \frac{\partial x^\beta}{\partial x'^{\,\sigma}} g_{\alpha\beta} \right) dx'^{\,\gamma} dx'^{\,\sigma} \\
&= \left(\frac{\partial x^\alpha}{\partial x'^{\,\mu}} \frac{\partial x^\beta}{\partial x'^{\,\nu}} g_{\alpha\beta} \right) dx'^{\,\mu} dx'^{\,\nu} \\
\Rightarrow \quad 0 &= \left(\frac{\partial x^\alpha}{\partial x'^{\,\mu}} \frac{\partial x^\beta}{\partial x'^{\,\nu}} g_{\alpha\beta} - g'_{\mu\nu} \right) dx'^{\,\mu} dx'^{\,\nu}
\end{aligned}
$$

$$
0 = \left(\frac{\partial x^\alpha}{\partial x'^{\,\mu}} \frac{\partial x^\beta}{\partial x'^{\,\nu}} g_{\alpha\beta} - g'_{\mu\nu} \right) \quad \Rightarrow \quad g'_{\mu\nu} = \frac{\partial x'^{\,\alpha}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x'^{\,\beta}} g_{\alpha\beta}
$$

$$
g^{\mu\nu} g_{\mu\alpha} = \delta^\nu{}_\alpha
$$

Tensors in a coordinate basis

Other tensors and tensor operations (which you can prove in a similar way):

The Kronecker delta: $\delta^{\nu}{}_{\alpha}$

The inverse metric: $g^{\mu\nu} g_{\mu\alpha} = \delta^{\nu}{}_{\alpha}$

Lowering an index: $A_{\mu} = g_{\mu\nu} A^{\nu}$

Raising an index: $B^{\mu} = g^{\mu\nu} B_{\nu}$

Gradients

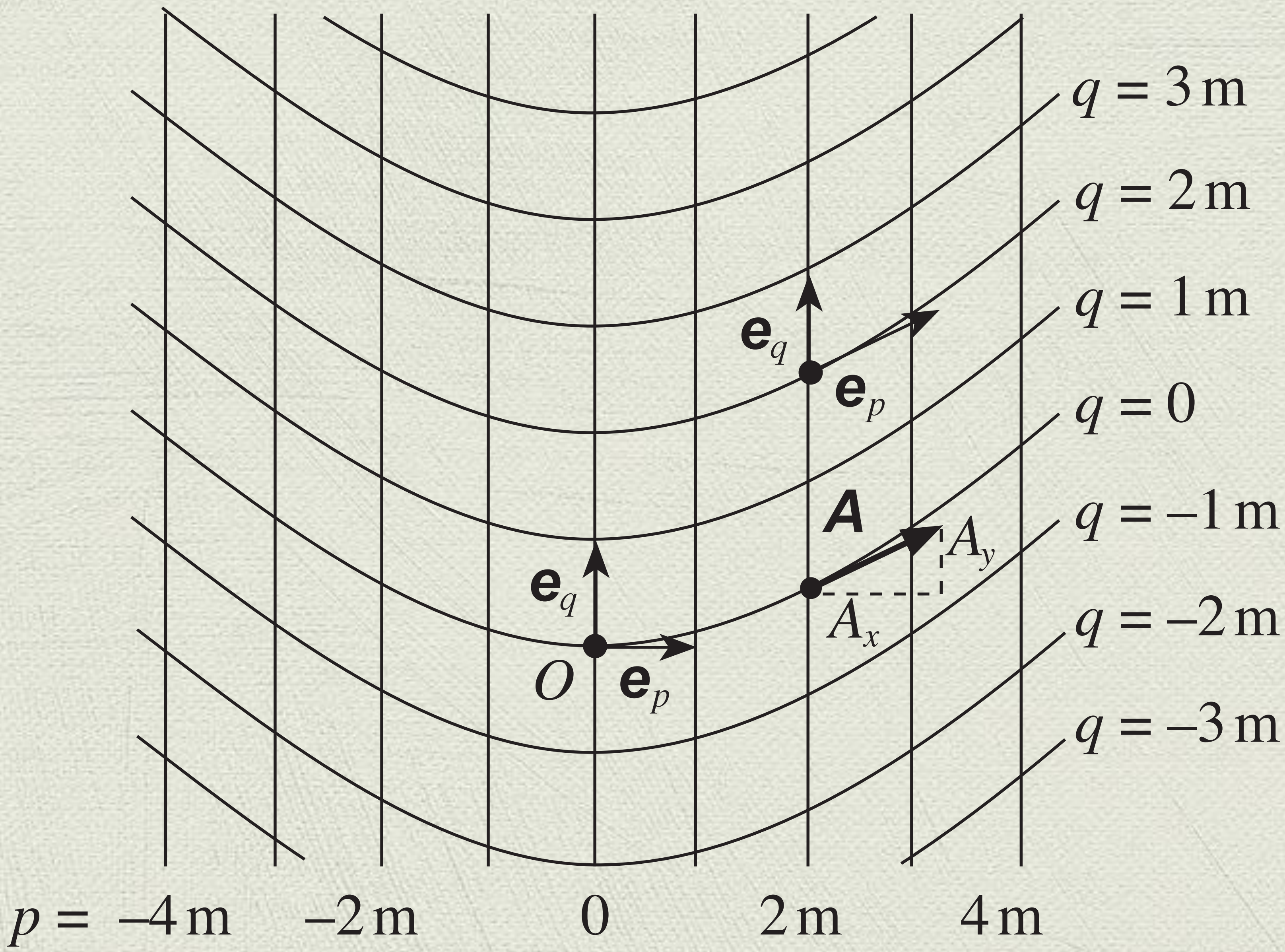
The gradient of a scalar is a covector:

$$\partial'_\mu \Phi \equiv \frac{\partial \Phi}{\partial x'^{\,\mu}} = \frac{\partial x^\nu}{\partial x'^{\,\mu}} \frac{\partial \Phi}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^{\,\mu}} (\partial_\nu \Phi)$$

The gradient of a vector is NOT a tensor:

$$\begin{aligned} \partial'_\mu A^\alpha \equiv \frac{\partial A^\alpha}{\partial x'^{\,\mu}} &= \frac{\partial}{\partial x'^{\,\mu}} \left(\frac{\partial x'^{\,\alpha}}{\partial x^\nu} A^\nu \right) = \frac{\partial x^\beta}{\partial x'^{\,\mu}} \frac{\partial}{\partial x^\beta} \left(\frac{\partial x'^{\,\alpha}}{\partial x^\nu} A^\nu \right) \\ &= \frac{\partial x^\beta}{\partial x'^{\,\mu}} \left(\frac{\partial^2 x'^{\,\alpha}}{\partial x^\beta \partial x^\nu} A^\nu \right) + \frac{\partial x^\beta}{\partial x'^{\,\mu}} \frac{\partial x'^{\,\alpha}}{\partial x^\nu} \left(\frac{\partial A^\nu}{\partial x^\beta} \right) \end{aligned}$$

Exercise: Parabolic coordinates



$$p(x, y) = x$$

$$q(x, y) = y - cx^2$$

Exercise: Parabolic coordinates

$$p(x,y) = x \qquad\qquad q(x,y) = y - cx^2$$

(a) Show that the inverse transformation functions are

$$x(p,q) = p \quad \text{and} \quad y(p,q) = cp^2 + q \tag{2.15}$$

(b) Evaluate all eight partial derivatives $\partial x'^{\,\mu}/\partial x^{\nu}$ and $\partial x^{\mu}/\partial x'^{\,\nu}$.

(c) The metric tensor components for cartesian coordinates in space are $g_{xx} = g_{yy} = 1, g_{xy} = g_{yx} = 0$. Use the general tensor transformation rule to show that the metric tensor for p, q coordinates is

$$g'_{\mu\nu} = \begin{bmatrix} 1 + 4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix} \tag{2.16}$$

(d) Let a vector $\boldsymbol{A}$ have components $A^p = 1, A^q = 0$ in the p, q coordinate system. Find this vector's components in the x, y coordinate system (as a function of x and y). But show that $A^2 = \boldsymbol{A} \cdot \boldsymbol{A}$ has the same value in both coordinate systems at every position.

Tensor gradient of a vector

Define Christoffel symbols:

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\mu} \equiv \Gamma^\nu_{\mu\alpha} \mathbf{e}_\nu$$

The physical change in a vector is then:

$$d\boldsymbol{A} = d(A^\mu \mathbf{e}_\mu) = \left(\frac{\partial A^\mu}{dx^\sigma} dx^\sigma \right) \mathbf{e}_\mu + A^\mu \frac{\partial \mathbf{e}_\mu}{\partial x^\alpha} dx^\alpha$$

$$= \left[\frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\nu} A^\nu \right] \mathbf{e}_\mu dx^\alpha \equiv (\nabla_\alpha A^\mu) \mathbf{e}_\mu dx^\alpha$$

So this **tensor gradient** is a tensor

$$\nabla_\alpha A^\mu \equiv \frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\nu} A^\nu$$

Tensor gradient of a covector

Tensor gradient of a scalar = ordinary gradient, so

$$\partial_\alpha (A^\mu B_\mu) = \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha}$$
$$= \nabla_\alpha (A^\mu B_\mu) = (\nabla_\alpha A^\mu) B_\mu + A^\mu (\nabla_\alpha B_\mu)$$

Subtracting and substituting yields, for arbitrary A^μ

$$0 = \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha} - (\nabla_\alpha A^\mu) B_\mu - A^\mu (\nabla_\alpha B_\mu)$$
$$= \cancel{\frac{\partial A^\mu}{\partial x^\alpha}} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha} - \cancel{\frac{\partial A^\mu}{\partial x^\alpha}} B_\mu - (\Gamma^\mu_{\alpha\nu} A^\nu) B_\mu - A^\mu (\nabla_\alpha B_\mu)$$
$$= A^\mu \left[\frac{\partial B_\mu}{\partial x^\alpha} - \Gamma^\nu_{\alpha\mu} B_\nu - \nabla_\alpha B_\mu \right]$$

So we must have: $\nabla_\alpha B_\mu = \dfrac{\partial B_\mu}{\partial x^\alpha} - \Gamma^\nu_{\alpha\mu} B_\nu$

Tensor gradient of a tensor

One Christoffel term for each index:

$$\nabla_\alpha T^{\mu\nu}{}_{\sigma} = \frac{\partial T^{\mu\nu}{}_{\sigma}}{\partial x^\alpha} + \Gamma^{\mu}_{\alpha\beta} T^{\beta\nu}{}_{\sigma} + \Gamma^{\mu}_{\alpha\delta} T^{\mu\delta}{}_{\sigma} - \Gamma^{\gamma}_{\alpha\sigma} T^{\mu\nu}{}_{\gamma}$$

Calculating Christoffel symbols:

We *assume* this symmetry (spacetime is “torsion-free”):

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$$

Then one can prove that:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left[\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right]$$

Exercise:

2.4.1 Exercise: The Tensor Gradient in Parabolic Coordinates.

Consider the parabolic coordinate system described in the previous exercise, where $p(x, y) = x$ and $q(x, y) = y - cx^2$. Consider a truly constant vector field in Cartesian coordinates defined so that $H^x = 1$ and $H^y = 0$.

(a) Find the components of $\boldsymbol{H}$ in the p, q coordinate system.

(b) In the previous exercise, we found the metric tensor for p, q coordinates to be

$$g'_{\mu\nu} = \begin{bmatrix} 1 + 4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix} \tag{2.30}$$

Verify (by matrix multiplication) that the inverse metric is given by

$$g'^{\mu\nu} = \begin{bmatrix} 1 & -2cp \\ -2cp & 1 + 4c^2p^2 \end{bmatrix} \tag{2.31}$$

(c) The Christoffel symbols for p, q coordinates are

$$\Gamma^q_{pp} = 2c, \quad \text{all other } \Gamma^\alpha_{\mu\nu} = 0 \tag{2.32}$$

(This makes sense, because we can see from Figure 2 that only the $\boldsymbol{e}_p$ unit vector changes with position, and then only in the q direction as we vary p.) Use $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right]$ to verify that these are the correct values for Γ^q_{pp} and Γ^p_{pp}.

(d) Calculate all four components of $\nabla_\mu H^\nu$ in p, q coordinates. Are the results what you expect?

The Geodesic Equation

A geodesic is *locally straight:*

$$0 = \frac{d\boldsymbol{u}}{d\tau} = \frac{d}{d\tau}(u^{\mu}\boldsymbol{e_{\mu}}) = \frac{du^{u}}{d\tau}\boldsymbol{e}_{\mu} + u^{\mu}\frac{d\boldsymbol{e}_{\mu}}{d\tau}$$

$$0 = \frac{d^2x^u}{d\tau^2}\boldsymbol{e}_{\mu} + \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\frac{d\boldsymbol{e}_{\mu}}{dx^{\nu}} = \frac{d^2x^u}{d\tau^2}\boldsymbol{e}_{\mu} + \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\mathit{\Gamma}^{\alpha}_{\mu\nu}\boldsymbol{e}_{\alpha}$$

$$= \left[\frac{d^2x^u}{d\tau^2} + \mathit{\Gamma}^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}\right]\boldsymbol{e}_{\mu}$$

Therefore, the geodesic equation is:

$$0 = \frac{d^2x^u}{d\tau^2} + \mathit{\Gamma}^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}$$

Exercise

Consider the parabolic coordinate system for two-dimensional flat space described in the previous two exercises, where $p(x, y) = x$ and $q(x, y) = y - cx^2$. In a two dimensional space like this, we parameterize paths using the arclength s along the path instead of the proper time, so the geodesic equation becomes

$$0 = \frac{d^2x^u}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} \tag{2.32}$$

and we look for solutions of the form $x^{\mu}(s)$. We found in the last exercise that the Christoffel symbols for this coordinate system all zero except for $\Gamma^{q}_{pp} = 2c$.

(a) Find the p and q components of the geodesic equation.

(b) The solution to the p-component equation is easy: $p = as$, where a is a constant of integration, if we define s to be zero where p is zero. Use this to show that the solution to the q-component equation is $q = -ca^2s^2 + bs + q_0$, where b and q_0 are constants of integration.

(c) The transformations back to cartesian coordinates are $x(p, q) = p$ and $y(p, q) = cp^2 + q$. Use these transformations to convert the solutions for $p(s)$ and $q(s)$ to solutions for $x(s)$ and $y(s)$. Argue that the resulting solutions are straight lines (*Hint:* Express y as a function of x.)

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

meaning that

$$g_{tt} = -(1 - 2GM/r), g_{rr} = (1 - 2GM/r)^{-1}, g_{\theta\theta} = r^2 \text{ and } g_{\phi\phi} = r^2 \sin^2\theta$$

and other metric components are zero.

Example:
four-velocity of a particle at rest

Remember the magnitude of a four-vector is –1:

$$\boldsymbol{u}\cdot\boldsymbol{u} \equiv u^\mu g_{\mu\nu} u^\nu = \frac{dx^\mu}{d\tau} g_{\mu\nu} \frac{dx^\nu}{d\tau} = \frac{g_{\mu\nu}dx^\mu dx^\nu}{d\tau^2} = \frac{ds^2}{d\tau^2} = \frac{-d\tau^2}{d\tau^2} = -1$$

This allows us to find u^t for a particle at rest:

$$-1 = u^\mu g_{\mu\nu} u^\nu = g_{tt}(u^t)^2 + 0 + 0 + 0$$

$$\Rightarrow \quad u^t = \frac{dt}{d\tau} = \sqrt{\frac{1}{-g_{tt}}} = \frac{1}{\sqrt{1 - 2GM/r}}$$

Example:
geodesic for a particle initially at rest

So the r-component of the geodesic equation is:

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma^r_{tt}u^t u^t + \text{zeros}$$

$$\Gamma^r_{tt} = \tfrac{1}{2}g^{r\alpha}(\partial_t g_{t\alpha} + \partial_t g_{\alpha t} - \partial_\alpha g_{tt}) = \tfrac{1}{2}g^{rr}(0 + 0 - \partial_r g_{tt})$$
$$= -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)\frac{d}{dr}\left(-1 + \frac{2GM}{r}\right) = +\left(1 - \frac{2GM}{r}\right)\frac{GM}{r^2}$$

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt}u^t u^t = -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)\frac{1}{1 - 2GM/r} = -\frac{GM}{r^2}$$

Exercise

It turns out one solution to the full geodesic equation for a radially falling particle is $dr/d\tau = -\sqrt{2GM/r}$. At what value of r is the particle at rest? At what value of r is $dr/d\tau = 1$? Does that necessarily mean that it is traveling at the speed of light?

Local Flatness Theorem

At any given event $\mathcal{P}$ in spacetime, our freedom to choose coordinates allows us to construct a coordinate system where (1) $g_{\mu\nu}(\mathcal{P}) = \eta_{\mu\nu}$ (with six degrees of freedom left over that correspond to arbitrary rotations and Lorentz boosts), (2) all 40 of the independent values of $\partial_\alpha g_{\mu\nu}|_\mathcal{P} = 0$, and (3) all but 20 of the 100 independent values of $\partial_\alpha \partial_\beta g_{\mu\nu}|_\mathcal{P}$ equal to zero.

Finding Components in a LOF

Assume that we know basis vectors of a LOF. Since:

$$\boldsymbol{A} = A^{\nu}\boldsymbol{e}'_{\nu} \text{ and } g'_{\mu\nu} \equiv \boldsymbol{e}'_{\mu} \cdot \boldsymbol{e}'_{\nu},$$

$$\boldsymbol{e}'_{\mu} \cdot \boldsymbol{A} = \boldsymbol{e}'_{\mu} \cdot \boldsymbol{e}'_{\nu} A'^{\,\nu} = g'_{\mu\nu} A'^{\,\nu} = A'_{\mu}$$

implying that

$$\eta^{\alpha\mu}(\boldsymbol{e}'_{\mu} \cdot \boldsymbol{A}) = \eta^{\alpha\mu} A'_{\mu} = A'^{\,\alpha}$$

Example: an observer at rest in Schwarzschild spacetime

$\boldsymbol{o}_T = \boldsymbol{u}_{\text{obs}}$. We know from before that $u^t_{\text{obs}} = \dfrac{1}{\sqrt{-g_{tt}}} = \dfrac{1}{\sqrt{1 - 2GM/r}}$

We can choose the spatial basis vectors to be normalized versions of the spatial Schwarzschild basis vectors. In particular:

$$1 = \boldsymbol{o}_z \cdot \boldsymbol{o}_z = g_{\mu\nu}(\boldsymbol{o}_z)^\mu(\boldsymbol{o}_z)^\nu = g_{rr}(\boldsymbol{o}_z)^r(\boldsymbol{o}_z)^r$$

$$\Rightarrow \quad (\boldsymbol{o}_z)^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{1 - 2GM/r}$$

Example: an observer at rest in Schwarzschild spacetime

A particle falling at rest from infinity: $u^r = dr/d\tau = -\sqrt{2GM/r}$

$$
\begin{aligned}
-1 &= g_{tt}(u^t)^2 + g_{rr}(u^r)^2 \\
&= -\left(1 - \frac{2GM}{r}\right)(u^t)^2 + \left(\frac{1}{1 - 2GM/r}\right)\frac{2GM}{r}
\end{aligned}
$$

$$
\Rightarrow -\left(1 - \cancel{\frac{2GM}{r}}\right) = -\left(1 - \frac{2GM}{r}\right)^2 (u^t)^2 + \cancel{\frac{2GM}{r}}
$$

$$
\Rightarrow \quad u^t = \frac{1}{1 - 2GM/r}
$$

Example: an observer at rest in Schwarzschild spacetime

Summary:

$$[\,(\boldsymbol{o}_T)^t, (\boldsymbol{o}_T)^r, (\boldsymbol{o}_T)^\theta, (\boldsymbol{o}_T)^\phi\,] = \left[\frac{1}{\sqrt{1-2GM/r}},\ 0,\ 0,\ 0\right]$$

$$[\,(\boldsymbol{o}_z)^t, (\boldsymbol{o}_z)^r, (\boldsymbol{o}_z)^\theta, (\boldsymbol{o}_z)^\phi\,] = \left[\ 0, \sqrt{1-2GM/r},\ 0,\ 0\ \right]$$

$$[\,u^t, u^r, u^\theta, u^\phi\,] = \left[\frac{1}{1-2GM/r},\ -\frac{2GM}{r},\ 0,\ 0\right]$$

Example: an observer at rest in Schwarzschild spacetime

Calculate the LOF components in Schwarzschild coordinates:

$$\begin{aligned}
u^T &= \eta^{T\mu}\boldsymbol{o}_\mu \cdot \boldsymbol{u} = \eta^{TT}\boldsymbol{o}_T \cdot \boldsymbol{u} = (-1)(\boldsymbol{o}_T)^\alpha g_{\alpha\beta}u^\beta = -(\boldsymbol{o}_T)^t g_{tt}u^t \\
&= +\frac{1}{\sqrt{1-2GM/r}}\left(1-\frac{2GM}{r}\right)\frac{1}{1-2GM/r} = \frac{1}{\sqrt{1-2GM/r}} \\
u^z &= \eta^{z\mu}\boldsymbol{o}_\mu \cdot \boldsymbol{u} = \eta^{zz}\boldsymbol{o}_z \cdot \boldsymbol{u} = (+1)(\boldsymbol{o}_z)^\alpha g_{\alpha\beta}u^\beta = +(\boldsymbol{o}_z)^r g_{rr}u^r \\
&= -\sqrt{\frac{2GM}{r}}\frac{1}{1-2GM/r}\sqrt{1-2GM/r} = \frac{-2GM/r}{\sqrt{1-2GM/r}}
\end{aligned}$$

Example: an observer at rest in Schwarzschild spacetime

Now we can calculate the falling particle's speed in the LOF:

$$v_x = \frac{u^x}{u^T} = 0, \quad v_y = \frac{u^y}{u^T} = 0,$$

$$v_z = \frac{u^z}{u^T} = \frac{-2GM/r}{\sqrt{1-2GM/r}}\sqrt{1-2GM/r} = -\frac{2GM}{r}$$

Summary: general method for calculating quantities in a LOF/LIF

1. Use the observer's four-velocity $\boldsymbol{u}_{\text{obs}}$ as the the observer's time-directed basis vector $\boldsymbol{o}_T$.
2. Construct a set of spatial basis vectors $\boldsymbol{o}_x, \boldsymbol{o}_y, \boldsymbol{o}_z$ such that $\boldsymbol{o}_\mu \cdot \boldsymbol{o}_\nu = \eta_{\mu\nu}$.
3. Find the components of $\boldsymbol{o}_\mu$ in whatever global coordinate system describes spacetime on the large scale.
4. Also determine the components of the four-vector $\boldsymbol{A}$ of interest in that global coordinate system.
5. The components of the four-vector in the observer's system are $A'^\mu = \eta^{\mu\nu} \boldsymbol{o}_\nu \cdot \boldsymbol{A}$, where one evaluates the dot product $\boldsymbol{o}_\mu \cdot \boldsymbol{A} = g_{\alpha\beta}(\boldsymbol{o}_\mu)^\alpha A^\beta$ in the *global* coordinate system.