General Relativity and Gravitational Waves

Session 3: The Einstein Equation

3.1 Overview of this Session

In the last session, we learned how to write tensor equations that apply in arbitrary coordinate systems (at least those with coordinate bases) and we also learned how to calculate geodesics in an arbitrary spacetime, one of the two core equations of general relativity. In this session, we will use tensor mathematics to construct the other core equation, which we call the Einstein Equation. The process that we use will be similar to the process that we used in the first session to derive Maxwell's equations.

An overview of this session's sections follows:

3.2 The Stress-Energy Tensor. In this section, we will argue that the source of the gravitational field is not a four-vector (as is the source of the electromagnetic field) but rather a second-rank tensor $\boldsymbol{T}$ we call the **stress-energy tensor,** and develop some expressions for this tensor in certain cases. We will also see how the tensor divergence of this field expresses local conservation of energy.

3.3 The Riemann Tensor. This section will explore the **Riemann Tensor**, a tensor quantity that is zero in flat spacetime but has 20 independent nonzero components in a curved spacetime. We will learn how to construct such a tensor and describe the built-in symmetries that constrain the number of independent components that it has.

3.4 Constructing the Einstein Equation. Here we will construct the simplest self-consistent tensor equation to describe how the curvature of spacetime can be connected to its source (the stress-energy tensor) in a way that is consistent with local conservation of energy.

3.5 Is Local Energy Conservation Geometrically Necessary? This section looks at the Einstein Equation from a different perspective, arguing that we can consider local energy conservation a consequence of the arbitrary nature of coordinates rather than being an external condition that we impose on the Einstein Equation. The section also argues that the geodesic equation itself follows from local energy conservation, meaning that theory really has only *one* core equation!

3.6 Does a Gravitational Field Have Energy? This section examines some vexing problems associated with any kind of attempt to calculate a local energy density for the gravitational field, and indeed with any kind of global concept of energy conservation in general relativity.

3.2 The Stress-Energy Tensor.

In the first section, in the context of "deriving" Maxwell's equations, we sought to find a tensor generalization of the Poisson equation $-\nabla^2\phi = \rho/\varepsilon_0$, which expresses Gauss's law in the context of a static electric field. Our first step involved determining what kind of tensor quantity has the charge density ρ, deciding eventually that it must be the time component of a four-vector.

Similarly, our task in this section is to find a tensor generalization of the Newtonian Poisson equation for gravity, which is $\nabla^2\phi = 4\pi G\rho$, where G is Newton's gravitational constant and ρ here is mass density. Our first task is analogously to determine what kind of tensor quantity has ρ as a component.

Part of what we need to resolve is whether ρ is this situation is actually *mass* density or *energy* density. In Newtonian situations (even in the fiery heart of the sun), particles' kinetic energies are so small that the energy of those particles is virtually entirely rest energy, so the distinction between mass density and energy density is negligible.

But can we resolve this question theoretically? Consider a small mirrored box that contains a positron and an electron bouncing around inside it. These particles will create a (very tiny) gravitational field outside the box, and suppose that this field accelerates a particle outside the box toward the box's center, increasing the latter particle's kinetic energy. But suppose that just as the particle is passing the box, the electron and positron meet and annihilate, creating two photons, which subsequently still bounce around inside the mirrored box.

If the source of the gravitational field was the positron and electron's rest mass, then because that mass is now gone, the gravitational field vanishes. The particle then passes the box and carries the kinetic energy it gained out to infinity without losing any of it due to the effects of the now nonexistent gravitational field. Therefore, we will have created energy (the particle's kinetic energy) from nothing (the box still contains the same total energy as it did initially).

On the other hand, if the source of the gravitational field is energy, then since the box's energy is conserved, its gravitational field is constant. Therefore the particle does not gain any net kinetic energy from its gravitational encounter with the box, and the paradox is avoided.This argument is by no means a rigorous proof, but it does make make plausible the idea that we should choose energy over mass as being the source of the gravitational field.

The Stress-Energy of Dust. Now we can turn our attention to what tensor quantity might represent the energy density ρ. Consider first a very simple fluid that consists of identical particles that (at least within a certain small neighborhood of a given event $\mathcal{P}$ have the same four-velocity. We call this fluid model **dust**, because the particles behave like airborne dust particles exposed by a shaft of sunlight: the dust particles in a given small region move along together in the direction of the ambient airflow.

The dust model means that at every event, we can find a locally inertial reference frame (LIF) in which the particles in a given small volume are essentially at rest. Consider N identical particles, each with mass m at rest inside a tiny box of volume V_0. The number density of particles in the box is thus $n_0 = N/V_0$, and since a particle's energy at rest is simply its mass m, the total energy density is $\rho_0 = n_0 m$.

Suppose that we now view this box in a different LIF centered at the same event but in which the box and all the particles inside move together with an ordinary velocity $\vec{v}$. The box as viewed in this new frame still has N particles in it (as every observer who can count will agree) but the box's volume is now $V = V_0\sqrt{1-v^2}$ because the box is Lorentz-contracted by that factor. The number density of the particles in the box in this frame is therefore $n = N/V = N/V_0\sqrt{1-v^2} = n_0/\sqrt{1-v^2}$.

Now, note that the particles' four-velocity in this frame (assuming the LIF uses t, x, y, z coordinates) is

$$u^\mu = \begin{bmatrix} 1/\sqrt{1-v^2} \\ v_x/\sqrt{1-v^2} \\ v_y/\sqrt{1-v^2} \\ v_z/\sqrt{1-v^2} \end{bmatrix} = \begin{bmatrix} u^t \\ v_x u^t \\ v_y u^t \\ v_z u^t \end{bmatrix} \tag{3.1}$$

Therefore, we can write the particle number density in this frame in the form

$$n = \frac{N}{V} = \frac{N}{V_0\sqrt{1-v^2}} = n_0 u^t \tag{3.2}$$

Since the energy of each particle in this frame is $p^t \equiv m u^t$ the total energy density measured in this frame is

$$\rho \equiv n p^t = (n_0 u^t)(m u^t) = (n_0 m) u^t u^t = \rho_0 u^t u^t \tag{3.3}$$

Now the energy density ρ_0 of the dust in its own frame is a relativistic scalar, because all observers know which LIF is at rest with respect to the dust and what an observer in that frame will measure. So the quantity in the equation above is a scalar multiple by the time components of two four-vectors. We see that in this case, the energy density of the fluid is the time-time component of the second-rank tensor

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \quad \text{(for dust)} \tag{3.4}$$

Note that this tensor is symmetric: $T^{\mu\nu} = T^{\nu\mu}$

The Components of $\boldsymbol{T}$. What are the other components of this tensor? We can write T^{tx} in the form

$$T^{tx} = \rho_0 u^t u^x = (n_0 m) u^t u^x = (n_0 u^t) m u^x = x\text{-momentum density} \tag{3.5}$$

Similarly, T^{ty} and T^{tz} correspond to y- and z-momentum density, respectively. Since $\boldsymbol{T}$ is symmetric, the same interpretations also apply to T^{xt}, T^{yt}, and T^{zt}.

However, we can also look at component T^{tx} in a different way:

$$T^{tx} = \rho_0 u^t u^x = (n_0 m) u^t u^x = (n_0 u^t) m (u^t v_x) = n p^t v_x = \frac{(n A v_x\, dt) p^t}{A dt} \tag{3.6}$$

The quantity $A v_x\, dt$ is the volume of dust that will move through an area A perpendicular to the x direction during the time interval dt, so $n A v_x\, dt$ is the number of particles that move through that area during that time. Therefore we can also think of $T^{tx} = T^{xt}$ as specifying the total energy per unit area per unit time (that is, the **energy flux**) flowing through a surface perpendicular to the x direction. Similarly, we can interpret $T^{ty} = T^{yt}$ and $T^{tz} = T^{zt}$ as representing energy fluxes in the y and z directions.

We can interpret T^{xy} in a similar way:

$$T^{xy} = \rho_0 u^x u^y = (n_0 m) u^x (u^t v_y) = (n_0 u^t)(m u^x) v_y = n p^x v_y = \frac{(nAv_y\,dt)p^x}{Adt} \tag{3.7}$$

that is, as flux of x-momentum through a surface perpendicular to the y direction (or by simply reordering the terms, the flux of y-momentum through a surface perpendicular to the x direction). We can interpret the other spatial components of $\boldsymbol{T}$ analogously.

What do these fluxes mean physically? Suppose we have a plate of area A perpendicular to the x direction in the path of the dust particles, and suppose that the plate absorbs those particles (and thus their energy and momentum). The quantity $T^{yx}A$ yields the *y-momentum per unit time* deposited on the plate (as measured in the LIF), which is the definition of the *y-component of the force* the dust exerts on the plate. **Stress** in physics is defined to be the force applied per unit area (this concept is more general than "pressure" because we don't require the force in a stress to act in a direction perpendicular to the plate). The components of $\boldsymbol{T}$ with two spatial indices therefore describe *stresses*, while the *tt*-component describes *energy* density. This is why we call $\boldsymbol{T}$ the **stress-energy tensor**.

Now, dust is a simple but not particularly realistic fluid, because the particles in most realistic fluids move with random thermal velocities that can be quite large relative to the fluid's bulk velocity. In a fluid with no viscosity (which we call a **perfect fluid**) we can treat the fluid in a given neighborhood as the sum over sets of dust particles, with each set consisting of the particles in that neighborhood that have roughly the same velocity. Because we are just adding dust stress-energies, the total tensor is still symmetric and we can interpret its components in the same way. In a LIF where the fluid in the neighborhood has no bulk velocity, the various sets have random velocities that do not favor any coordinate direction, so there is no net density of x-, y- or z-momentum. The net flows of x-momentum in y and z directions will also cancel out, as do net flows of y-momentum in the x and z directions and net flows of z-momentum in the x and y directions. However, the sum of dust tensor components like $\rho_0 u^x u^x = \rho_0 (u^t v_x)(u^t v_x) = \rho v_x^2$ do *not* cancel out, because the result is always positive. So the diagonal spatial components will add up to something nonzero, but since no direction is preferred in random motion, the sums will have to be the same. So in a LIF where the fluid is at rest at the LIF's origin, the total stress-energy at that event will be something like

$$T^{\mu\nu} = \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{bmatrix} \quad \text{(for a perfect fluid at rest in a LIF)} \tag{3.8}$$

where ρ_0 is now the density of the total fluid and p_0 is now the pressure of the fluid, both as evaluated in the LIF where the fluid is at rest at the LIF's origin. These are both relativistic scalars (because every observer will agree which LIF is at rest with respect to the fluid at any given point and what observers in that LIF measure), but their values may change with position in spacetime.

Now consider the following tensor expression in arbitrary coordinates:

$$T^{\mu\nu} = (\rho_0 + p_0) u^\mu u^\nu + p_0\, g^{\mu\nu} \tag{3.9}$$

This is clearly a tensor expression, but in a LIF where the fluid is at rest (so that its four-velocity components are $u^t = 1, u^x = u^y = u^z = 0$), the components of this tensor become

$$T^{tt} = (\rho_0 + p_0) u^t u^t + p_0\, \eta^{tt} = (\rho_0 + p_0) - p_0 = \rho_0 \tag{3.10}$$

$$T^{xx} = (\rho_0 + p_0) u^x u^x + p_0\, \eta^{xx} = 0 + p_0 = p_0 \quad \text{(and similarly for } T^{yy} \text{ and } T^{zz}\text{)} \tag{3.11}$$

$$T^{tx} = (\rho_0 + p_0) u^t u^x + p_0\, \eta^{tx} = 0 + 0 \quad \text{(and similarly for other off-diagonal components)} \tag{3.12}$$

So this tensor expression reduces to the form we inferred we should have at the origin of a LIF. Therefore, this is a general-coordinate expression for the stress-energy tensor for a perfect fluid.

This is still not a completely realistic model of a fluid, but for most practical applications, it is completely adequate. Indeed, the much cruder dust model is often adequate, because as we will see in an upcoming exercise, a fluid would need to be pretty relativistic to have a significant pressure relative to its energy density.

Now, just as $\partial_\mu J^\mu = 0$ expresses conservation of charge in the context of special relativity, the tensor expression

$$\nabla_\nu T^{\mu\nu} = 0 \tag{3.13}$$

expresses conservation of the fluid's energy and momentum in an arbitrary coordinate system. The argument is basically as follows. Consider evaluating this equation at the origin of a LIF. Since the Christoffel symbols are all zero at the origin of a LIF, the expression reduces to $\partial_\nu T^{\mu\nu} = 0$. Consider an infinitesimal box with dimensions dx, dy, dz centered on the LIF's origin event. Consider first the two faces of the box that are perpendicular to the x direction. Since T^{tx} is the energy flux in the x direction, the total energy flowing into the box through its left face during time dt is this flux times the face's area $dy\,dz$ times the time dt, that is, $(T^{tx})_{\text{left}} dy\,dz\,dt$, where $(T^{tx})_{\text{left}}$ refers to the tensor component evaluated at the left face. Similarly, the energy flowing out of the right face during the same time interval is $(T^{tx})_{\text{right}} dy\,dz\,dt$. The total energy that accumulates in the box due to particle flow in the x direction during this time interval is

$$\left[(T^{tx})_{\text{left}} - T^{tx})_{\text{right}}\right] dy\,dz\,dt = \left(-\frac{\partial T^{tx}}{\partial x}dx\right) dy\,dz\,dt = -\frac{\partial T^{tx}}{\partial x} dx\,dy\,dz\,dt \tag{3.14}$$

by definition of the partial derivative. The expressions for the net energy that accumulates due to flows in the other directions are analogous, so the total energy that accumulates in the box during that time interval is

$$dE = \left[-\frac{\partial T^{tx}}{\partial x} - \frac{\partial T^{ty}}{\partial y} - \frac{\partial T^{tz}}{\partial z}\right] dx\,dy\,dz\,dt \tag{3.15}$$

But assuming energy is conserved, any net energy that flows into the box through its sides must yield a net increase in the energy *inside* during that time dt, which is rate at which the energy density T^{tt}changes times dt times the volume of the box:

$$dE = \frac{\partial T^{tt}}{\partial t} dt\,dx\,dy\,dz \tag{3.16}$$

Subtracting the previous expression from this yields

$$0 = \left[\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{tx}}{\partial x} + \frac{\partial T^{ty}}{\partial y} + \frac{\partial T^{tz}}{\partial z}\right] dx\,dy\,dz\,dt \quad \Rightarrow \quad \partial_\nu T^{t\nu} = 0 \tag{3.17}$$

This equation therefore expresses the fact that energy is conserved. Similarly, $\partial_\nu T^{x\nu} = 0$ expresses conservation of the fluid's x-momentum, $\partial_\nu T^{y\nu} = 0$ expresses conservation of the fluid's y-momentum, and $\partial_\nu T^{z\nu} = 0$ expresses conservation of the fluid's z-momentum.

Now if a tensor is zero at an event in any coordinate system, it is zero in *all* coordinate systems by the basic tensor transformation rule. Since we have seen that the tensor $\nabla_\nu T^{\mu\nu}$ is zero at the origin of LIF (which we can always set up at an arbitrary event), it must be zero at all events in all coordinate systems.

3.2.1 Exercise: Perfect fluid pressures.

Pressure has units of force per unit area, which in units where we measure time and distance in meters, will be $(\text{kg m/m}^2)/\text{m}^2 = \text{kg/m}^3$ just like energy density $(\text{kg m}^2/\text{m}^2)/\text{m}^3$. What random speeds would particles in a fluid need to have if the fluid pressure is even 1% of the fluid density in these units? What approximate temperature would this correspond to if the particles are electrons? (*Hints:* Remember that the average of v_x^2 will be $\frac{1}{3}$ times the average of v^2. The Newtonian approximation for the electrons' kinetic energy is adequate as a first approximation, and remember that $kT \approx \frac{1}{40}$ eV at 300 K and that an electron's rest energy is about 0.5 MeV.)

3.3 The Riemann Tensor.

We now know what goes on the right side of our tensor generalization of the Newtonian field equation $\nabla^2\phi = 4\pi G\rho$. What about the left side? Here we need to do something a bit different than what we did in "deriving" Maxwell's equation. In the Maxwell case, we were looking for a vector field "on top" of a flat spacetime, and if we do a similar thing here, the natural result is a gravitational theory that represents the gravitational field as a tensor field "on top" of flat spacetime. But we are looking instead for a field that *defines the shape of spacetime itself.* This calls for a deeper approach that works from first principles.

In the first session, we saw that the fundamental nature of gravity is that it *curves* spacetime, making that spacetime non-Euclidean. So our first task is to find a tensor quantity that describes the curvature of spacetime, that is, a tensor that is zero if the spacetime is flat (indicating the absence of a gravitational field) and nonzero if it is curved.

Detecting whether a space or spacetime is curved or not is a non-trivial task. One can certainly not tell whether a spacetime is curved or not simply by looking at the metric. Consider some metrics that we have seen before for two-dimensional spaces:

$$g_{\mu\nu} = \begin{bmatrix} 1+4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix}, \qquad g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix} \tag{3.18}$$

There is no obvious clue in these metrics that the first is flat space in disguise, while the second describes an intrinsically curved space.

The most logical approach would be to go back to the most fundamental characteristic of a curved spacetime: the relative acceleration of neighboring geodesics. One can in fact use this approach (and I do in my textbook[1]), but the mathematics is quite involved and not particularly transparent. Time is short here, so I am going to present a different approach that takes advantage of the Local Flatness theorem. Consider the tensor quantity (where $\boldsymbol{A}$ is an arbitrary four-vector):

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A^\alpha = \nabla_\mu(\nabla_\nu A^\alpha) - \nabla_\nu(\nabla_\mu A^\alpha) \tag{3.19}$$

This quantity expresses the commutativity of the tensor gradient. Suppose that we evaluate this quantity in a truly flat spacetime. In such a spacetime, we can find a truly Euclidean coordinate system where not only the metric reduces to $\eta_{\mu\nu}$ but *all derivatives of the metric* are also zero. In such a case, the quantity above reduces to $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)A^\alpha = 0$, because the order of partial derivatives does not matter. But in a curved spacetime, the Local Flatness Theorem states that we can never set all of the metric's *second derivatives* to zero (the best we can do is set all but 20 equal to zero at the system's origin). But as we expand the expressions for the tensor gradients in the expression above, we will find that we are evaluating derivatives of Christoffel symbols, and such derivatives will involve second derivatives of the metric, which will *not* generally be zero in a curved spacetime. So this might be the tensor quantity we seek.

Let's evaluate this quantity in terms of Christoffel symbols to get more insight. Treating each of the terms in equation 3.19 separately and using the rules for the tensor gradient, we have

$$\begin{aligned} \nabla_\mu(\nabla_\nu A^\alpha) &= \partial_\mu(\nabla_\nu A^\alpha) - \Gamma^\beta_{\mu\nu}(\nabla_\beta A^\alpha) + \Gamma^\alpha_{\mu\sigma}(\nabla_\nu A^\sigma) \\ &= \partial_\mu(\partial_\nu A^\alpha + \Gamma^\alpha_{\nu\gamma}A^\gamma) - \Gamma^\beta_{\mu\nu}(\partial_\beta A^\alpha + \Gamma^\alpha_{\beta\delta}A^\delta) + \Gamma^\alpha_{\mu\sigma}(\partial_\nu A^\sigma + \Gamma^\sigma_{\nu\rho}A^\rho) \\ &= \partial_\mu\partial_\nu A^\alpha + (\partial_\mu\Gamma^\alpha_{\nu\gamma})A^\gamma + \Gamma^\alpha_{\nu\gamma}\partial_\mu A^\gamma - \Gamma^\beta_{\mu\nu}\partial_\beta A^\alpha - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\delta}A^\delta + \Gamma^\alpha_{\mu\sigma}\partial_\nu A^\sigma + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\rho}A^\rho \end{aligned} \tag{3.20}$$

The second term is the same except with μ and ν swapped. So subtracting it from the term above yields

$$\begin{aligned} &+\cancel{\partial_\mu\partial_\nu A^\alpha} + (\partial_\mu\Gamma^\alpha_{\nu\gamma})A^\gamma + \cancel{\Gamma^\alpha_{\nu\gamma}\partial_\mu A^\gamma} - \cancel{\Gamma^\beta_{\mu\nu}\partial_\beta A^\alpha} - \cancel{\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\delta}A^\delta} + \cancel{\Gamma^\alpha_{\mu\sigma}\partial_\nu A^\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\rho}A^\rho \\ &-\cancel{\partial_\nu\partial_\mu A^\alpha} - (\partial_\nu\Gamma^\alpha_{\mu\gamma})A^\gamma - \cancel{\Gamma^\alpha_{\mu\gamma}\partial_\nu A^\gamma} + \cancel{\Gamma^\beta_{\nu\mu}\partial_\beta A^\alpha} + \cancel{\Gamma^\beta_{\nu\mu}\Gamma^\alpha_{\beta\delta}A^\delta} - \cancel{\Gamma^\alpha_{\nu\sigma}\partial_\mu A^\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\rho}A^\rho \\ &= (\partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta})A^\beta \end{aligned} \tag{3.21}$$

where I have taken advantage of the commutativity of the partial derivative, the symmetry of the lower two indices of the Christoffel symbols, and the irrelevance of bound index names to make the cancellations indicated. In the last step, I have renamed the bound γ index in the first two terms and the bound ρ index in the last two terms to β. Since this difference is a tensor and $\boldsymbol{A}$ is a tensor, the quantity in parentheses must be a tensor. We call this tensor the **Riemann tensor:**

$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta} \tag{3.22}$$

Here is a mnemonic for this expression. Note that the μ and ν indices are the first two free lower indices in all the terms, and they appear in that order in the positive terms, but reversed in the negative terms. The products of the Christoffel symbols involve a sum over the last index of the first symbol and the upper index of the next. So remember: "3 to 4 is positive and twins bond inside," meaning that the 3rd and 4th indices of the Riemann tensor appear in that order in the positive terms (and in the opposite order in negative terms), and the bound indices in the terms involving two Christoffel symbols are the indices closest together.

Note that the order of the indices is very important in the Riemann tensor, one of the first tensors we have seen where this is really so. Indeed the Riemann tensor has important symmetry properties that describe what happens when we change the index order.

We can see directly from its definition that the Riemann tensor is antisymmetric in its last two indices:

$$R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu} \tag{3.23}$$

However, the tensor has some other symmetries that are easiest to see if we lower the first index:

$$R_{\alpha\beta\mu\nu} = g_{\alpha\gamma} R^{\gamma}{}_{\beta\mu\nu} \tag{3.24}$$

Then, we can write the other symmetries as

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \qquad \text{(antisymmetry in the first two indices)} \tag{3.25}$$

$$R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta} \qquad \text{(symmetry in the the index pairs)} \tag{3.26}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \qquad \text{(cyclic rotation of the final three indices)} \tag{3.27}$$

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0 \qquad \text{(cyclic rotation of the gradient and last two indices)} \tag{3.28}$$

The final symmetry is the **Bianchi identity**, which we will find very important in what follows.

The easiest way to *prove* these identities is to evaluate these tensor equations at the origin of a LIF. At that event in a LIF, the Christoffel symbols are all zero, but the derivatives of those symbols (which involve second derivatives of the metric) are not necessarily zero. So at the origin of a LIF, we have

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= g_{\alpha\gamma} R^{\gamma}{}_{\beta\mu\nu} = g_{\alpha\gamma}(\partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta}) \\ &= g_{\alpha\gamma}\partial_\mu[\,\tfrac{1}{2} g^{\gamma\sigma}(\partial_\nu g_{\beta\sigma} + \partial_\beta g_{\sigma\nu} - \partial_\sigma g_{\nu\beta})\,] - g_{\alpha\gamma}\partial_\nu[\,\tfrac{1}{2} g^{\gamma\sigma}(\partial_\mu g_{\beta\sigma} + \partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\mu\beta})\,] \end{aligned} \tag{3.29}$$

Now, at the origin in a LIF, only the second derivatives of the metric are possibly nonzero, so we can pull the $g^{\gamma\sigma}$ out in front of the derivatives to get

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \tfrac{1}{2} g_{\alpha\gamma} g^{\gamma\sigma}(\cancel{\partial_\mu\partial_\nu g_{\beta\sigma}} + \partial_\mu\partial_\beta g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\nu\beta} - \cancel{\partial_\nu\partial_\mu g_{\beta\sigma}} - \partial_\nu\partial_\beta g_{\sigma\mu} + \partial_\nu\partial_\sigma g_{\mu\beta}) \\ &= \tfrac{1}{2}\delta^{\sigma}_{\alpha}(\partial_\mu\partial_\beta g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\nu\beta} - \partial_\nu\partial_\beta g_{\sigma\mu} + \partial_\nu\partial_\sigma g_{\mu\beta}) \\ &= \tfrac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\alpha\mu}) \qquad \text{(only at a LIF origin!)} \end{aligned} \tag{3.30}$$

The mnemonic here is "inner togetherness is positive:" when the inner indices $\beta\mu$ appear together in either the partial derivatives or the metric, the term is positive (the two negative terms split these indices up). We can see that the Riemann tensor does indeed involve second derivatives of the metric!

All of the symmetries are easy to prove using this form of the Riemann tensor. For example, using the "inner togetherness" mnemonic, we see that

$$\begin{aligned} R_{\beta\alpha\mu\nu} &= \tfrac{1}{2}(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\nu\partial_\alpha g_{\beta\mu}) \\ &= -\tfrac{1}{2}(\partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\alpha g_{\beta\nu} - \partial_\nu\partial_\beta g_{\mu\alpha}) \\ &= -\tfrac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\alpha\mu}) \equiv -R_{\alpha\beta\mu\nu} \end{aligned} \tag{3.31}$$

since the metric is symmetric. The other proofs are similarly straightforward.

Now, one of the reasons these symmetries are relevant is that we can use them to count independent components of the Riemann tensor. The tensor formally has $4 \times 4 \times 4 \times 4 = 256$ components, but many of these components are zero and even more are not independent. Let's represent the index names abstractly by 0, 1, 2, and 3 (rather than something like t, x, y, z that would refer to a specific coordinate system) First we note that any component whose final indices or initial indices are the same is automatically zero: for example, $R_{0023} = -R_{0023}$ after switching the first two indices, but only zero can equal negative itself. So only name for the index pairs $\alpha\beta$ and $\mu\nu$ that might yield independent tensor components are the six pairs 01, 02, 03, 12, 13, and 23: all other pairs yield either components that are zero or are related by index reversal. Let's arrange these possibly independent values in a chart:

$$\begin{array}{cccccccc} & \mu\nu \rightarrow & 01 & 02 & 03 & 12 & 13 & 23 \\ \alpha\beta\downarrow & 01 & \underline{R_{0101}} & R_{0102} & R_{0103} & R_{0112} & R_{0113} & R_{0123} \\ & 02 & R_{0201} & \underline{R_{0202}} & R_{0203} & R_{0212} & R_{0213} & R_{0223} \\ & 03 & R_{0301} & R_{0302} & \underline{R_{0303}} & R_{0312} & R_{0313} & R_{0323} \\ & 12 & R_{1201} & R_{1202} & R_{1203} & \underline{R_{1212}} & R_{1213} & R_{1223} \\ & 13 & R_{1301} & R_{1302} & R_{1303} & R_{1312} & \underline{R_{1313}} & R_{1323} \\ & 23 & R_{2301} & R_{2302} & R_{2303} & R_{2312} & R_{2313} & \underline{R_{2323}} \end{array} \tag{3.32}$$

Now, of the 36 components on this chart, the 15 above underlined diagonal elements are the same as the 15 below, because $R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta}$. Therefore, only 6 + 15 = 21 components on this chart are independent.

Finally, the symmetry equation $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$ looks like it puts 256 conditions on components of the metric, but actually, on the basis of what we know already, this equation is identically zero if *any* two indices have the same name. For example, say that the second and fourth indices have the same name 1: then the equation claims that $R_{\alpha 1\mu 1} + R_{\alpha 11\mu} + R_{\alpha\mu 11} = 0$. But the first two terms cancel because of the antisymmetry of the last two indices, and the last term is also zero because because of the same issue. Therefore this equation simply states that $0 = 0$, which does say anything useful. The only component of this equation that says something new is therefore $R_{0123} + R_{0312} + R_{0213} = 0$. This puts one additional constraint on the Riemann tensor components, leaving 20 that are independent.

(The Bianchi identity is about the *derivatives* of Riemann tensor components, so it does not put any more constraints on the components themselves.)

These 20 components are linearly independent combinations of the 20 metric double-derivatives that we cannot force to zero by choosing coordinates in a curved spacetime. The fact that these counts are equal means that none of these metric derivatives is omitted or canceled out when we construct the Riemann tensor. This means that *any* deviation from flat spacetime should be registered by the Riemann tensor.

3.3.1 Exercise: The Space Behind Parabolic Coordinates is Flat.

In a two-dimensional space, only one Riemann tensor component is possibly independent: R_{0101}.

(a) Explain how and why all other components depend on this one (or are zero).

(b) We have seen that for the p, q parabolic coordinate system we have discussed in previous exercises, only one Christoffel symbol was nonzero $\Gamma^q_{pp} = 2c$, where c is a constant. Show that $R_{pqpq} = 0$ in this case, proving purely from the metric that parabolic coordinates must describe a flat space. For fast reference, the metric for parabolic coordinates is

$$g_{\mu\nu} = \begin{bmatrix} 1 + 4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix} \tag{3.33}$$

3.4 Constructing the Einstein Equation.

Our goal in this section is to find the appropriate tensor generalization of the Newtonian gravitational field equation $\nabla^2\Phi = 4\pi G\rho$ (where Φ is the Newtonian gravitational potential and ρ is mass density. We saw at the beginning of this session that the appropriate generalization of the right side is the stress-energy tensor $T^{\mu\nu}$, so the appropriate tensor generalization might look something like

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{3.34}$$

where κ is a scalar constant and $G^{\mu\nu}$ is a second-rank tensor that tells us something about the curvature of spacetime. Now, we have just seen that the Riemann tensor describes the curvature of spacetime, but we can't set a fourth-rank tensor equal to a second-rank tensor. Therefore $G^{\mu\nu}$ must be constructed out of the Riemann tensor in such a way as to leave only two free indices.

The number of indices is not the only constraint on $G^{\mu\nu}$. Since $T^{\mu\nu}$ is a symmetric tensor ($T^{\mu\nu} = T^{\nu\mu}$), the $G^{\mu\nu}$ must also be symmetric. We also know that $\nabla_\nu T^{\mu\nu} = 0$ because of local conservation of energy and momentum, so (because taking the tensor divergence of both sides of the equation should still yield a valid tensor equation) we must also have $\nabla_\nu G^{\mu\nu} = 0$ as well. Finally, we do want to construct $G^{\mu\nu}$ out of the Riemann tensor (using only linear factors of that tensor if possible) and the metric tensor, as only these tensors are directly connected to the curvature of spacetime.

An obvious candidate for $G^{\mu\nu}$ is the **Ricci tensor**, which we construct by contracting the Riemann tensor over its first and third indices:

$$R_{\beta\nu} \equiv R^{\alpha}{}_{\beta\alpha\nu} \tag{3.35}$$

The form of the Ricci tensor with both indices raised is

$$R^{\mu\nu} = g^{\mu\beta}g^{\nu\sigma}R^{\alpha}{}_{\beta\alpha\sigma} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma} \tag{3.36}$$

Because the Riemann tensor is symmetric under exchange of the two pairs of indices ($R_{\gamma\sigma\alpha\beta} = +R_{\alpha\beta\gamma\sigma}$), the Ricci tensor is symmetric:

$$R^{\nu\mu} = g^{\nu\beta}g^{\mu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma} = g^{\nu\beta}g^{\mu\sigma}g^{\alpha\gamma}R_{\gamma\sigma\alpha\beta} = g^{\nu\sigma}g^{\mu\beta}g^{\gamma\alpha}R_{\alpha\beta\gamma\sigma} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma} \equiv R^{\mu\nu} \tag{3.37}$$

where I started with equation 3.36, then I swapped the first pair of indices with the second pair in the Riemann tensor, then I renamed the bound indices $\beta \leftrightarrow \sigma$ and $\alpha \leftrightarrow \gamma$, then used the symmetry of the metric and commutativity of multiplication, and then finally, applied equation 3.36 again.

However, $\nabla_\nu R^{\mu\nu} \neq 0$ in general. We can see this most easily at the origin of a LIF, where

$$\nabla_\nu R^{\mu\nu} = \nabla_\nu(g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma}) = \partial_\nu[\, g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}\tfrac{1}{2}(\partial_\beta\partial_\gamma g_{\alpha\sigma} + \partial_\alpha\partial_\sigma g_{\beta\gamma} - \partial_\beta\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\sigma})\,] \tag{3.38}$$

But at the origin of a LIF, the first derivatives of the metric are all zero, so we can pull the metric factors out in front and allow the ∂_μ to act on the metric factors that already have derivatives:

$$\nabla_\nu R^{\mu\nu} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}\tfrac{1}{2}(\partial_\nu\partial_\beta\partial_\gamma g_{\alpha\sigma} + \partial_\nu\partial_\alpha\partial_\sigma g_{\beta\gamma} - \partial_\nu\partial_\beta\partial_\sigma g_{\alpha\gamma} - \partial_\nu\partial_\alpha\partial_\gamma g_{\beta\sigma}) \tag{3.39}$$

Now, in the last term, renaming $\gamma \leftrightarrow \sigma$ and $\nu \leftrightarrow \alpha$, makes that term equal to the negative of the second term (since the order of partial derivatives and metric factors is irrelevant), so these terms cancel, leaving

$$\nabla_\nu R^{\mu\nu} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}\tfrac{1}{2}(\partial_\nu\partial_\beta\partial_\gamma g_{\alpha\sigma} - \partial_\nu\partial_\beta\partial_\sigma g_{\alpha\gamma}) \tag{3.40}$$

But if I try renaming $\gamma \leftrightarrow \sigma$ in the new final term, I get

$$\nabla_\nu R^{\mu\nu} = \tfrac{1}{2}g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}\partial_\nu\partial_\beta\partial_\gamma g_{\alpha\sigma} - \tfrac{1}{2}g^{\mu\beta}g^{\nu\gamma}g^{\alpha\sigma}\partial_\nu\partial_\beta\partial_\gamma g_{\alpha\sigma} \tag{3.41}$$

The derivatives become the same, but the metric factors in front are not equivalent. For example, we note that in the final term, the final metric factor sums over the indices of the metric in the derivative, but none of the metric factors multiplying the first term do the same thing. So these two terms are not equivalent and do not cancel. This means that $\nabla_\nu R^{\mu\nu}$ is not necessarily zero at the origin of the LIF, and so cannot be counted on to be zero in all coordinate systems.

So $G^{\mu\nu}$ cannot simply be equal to $R^{\mu\nu}$. But what other terms might $G^{\mu\nu}$ contain that involve only the Riemann tensor and the metric? Note that the inverse metric $g^{\mu\nu}$ by itself is a symmetric second-rank tensor. We could also have a term proportional to $g^{\mu\nu}R$, where R is the **curvature scalar**:

$$R \equiv g^{\beta\nu}R_{\beta\nu} = g^{\beta\nu}R^{\alpha}{}_{\beta\alpha\nu} = g^{\beta\nu}g^{\alpha\mu}R_{\alpha\beta\mu\nu} \tag{3.42}$$

In fact, $R^{\mu\nu}, g^{\mu\nu}R$ and $g^{\mu\nu}$ are the only second-rank symmetric tensors that one can construct out of the Riemann tensor and the metric tensor that are linear in the Riemann tensor.[2] Let's see if we can construct a tensor of the form $R^{\mu\nu} + bg^{\mu\nu}R + \Lambda g^{\mu\nu}$ (where b and Λ are scalar constants) such that

$$\nabla_\nu(R^{\mu\nu} + bg^{\mu\nu}R + \Lambda g^{\mu\nu}) = 0 \tag{3.43}$$

identically, If we can find such a tensor, we will have found the simplest choice for something that could be proportional to the stress-energy tensor in a tensor generalization of $\nabla^2\Phi = 4\pi G\rho$. Occam's razor suggests that this will be the *correct* choice.

Note that $\nabla_\nu g^{\mu\nu} = 0$ automatically because *the tensor gradient of the metric is zero.* Again, we can see this most easily at the origin of a LIF, where

$$\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} = 0 \tag{3.44}$$

because the first derivatives of the metric are zero at the origin of a LIF by construction. If this tensor expression is zero at an arbitrary event in any coordinate system it is zero in every coordinate system, so $\nabla_\nu g^{\mu\nu} = 0$. Since Λ is a constant, we have $\nabla_\nu(\Lambda g^{\mu\nu}) = \Lambda\nabla_\nu g^{\mu\nu} = 0$ as well. So our problem reduces to finding a value of b such that

$$\nabla_\nu(R^{\mu\nu} + bg^{\mu\nu}R) = 0 \tag{3.45}$$

The key to solving for b is the Bianchi Identity (equation 3.28), repeated here for convenience:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0 \tag{3.46}$$

Consider multiplying both sides of this equation by $g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}$ and sum over the repeated index names:

$$g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}\nabla_\sigma R_{\alpha\beta\mu\nu} + g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}\nabla_\nu R_{\alpha\beta\sigma\mu} + g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}\nabla_\mu R_{\alpha\beta\nu\sigma} = 0 \tag{3.47}$$

Now, we have seen (equation 3.44) that the tensor gradient of the metric is zero, so the metric factors in the equation above behave like constants, so we can pull them inside the derivative in each case:

$$\nabla_\sigma(g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}) + \nabla_\nu(g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\sigma\mu}) + \nabla_\mu(g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\nu\sigma}) = 0 \tag{3.48}$$

Now, if you look at the first term carefully, you will see that the final two inverse metric factors summed with the Riemann tensor coincides with the definition of the curvature scalar R. In the second term, if we reverse the last two indices (which changes the term's sign) then the middle inverse metric factor sums over the first and third Riemann indices, making it the Ricci tensor, and the other two inverse metric factors simply raise the remaining indices. We can use more Riemann symmetries and some renaming (see the exercise) to show that the last term is the same as the second, implying that

$$\nabla_\sigma g^{\gamma\sigma}R - 2\nabla_\sigma R^{\gamma\sigma} = 0 \quad \Rightarrow \quad \nabla_\sigma(R^{\gamma\sigma} - \tfrac{1}{2}g^{\gamma\sigma}R) = 0 \tag{3.49}$$

identically. Therefore, choosing $b = -\frac{1}{2}$ does exactly what we want.

The most general form of the field equation for gravity should therefore be:

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \tag{3.50}$$

This is the **Einstein equation**, though not quite in its final form.

Before we go on, note that if we multiply both sides of equation 3.50 by $g_{\mu\nu}$ and do the sums, we get

$$g_{\mu\nu}R^{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\mu\nu}R + \Lambda g_{\mu\nu}g^{\mu\nu} = \kappa g_{\mu\nu}T^{\mu\nu} \tag{3.51}$$

Now, $g_{\mu\nu}R^{\mu\nu} \equiv R$, and $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$, because we are summing down the diagonal of the identity matrix. If we define $T \equiv g_{\mu\nu}T^{\mu\nu}$, this equation becomes

$$R - 2R + 4\Lambda = -R + 4\Lambda = \kappa T \tag{3.52}$$

If we multiply both sides of last equation by $\frac{1}{2}g^{\mu\nu}$, subtract it from equation 3.50, and move the Λ term over to the other side, we get

$$R^{\mu\nu} = \kappa(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}T) + \Lambda g^{\mu\nu} \tag{3.53}$$

This is actually the form of the Einstein equation that is easiest to solve, because it puts the complexity on the right-hand side (where it is easier to handle).

3.4.1 Exercise: Filling in the Gaps

(a) Explain why $\nabla_\mu(g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\nu\sigma}) = -\nabla_\sigma R^{\gamma\sigma}$. Specifically, describe exactly which symmetries of the Riemann tensor we have to use and how to rename indices to get this result.

(b) Fill in the gaps between equations 3.50, 3.52, and 3.53.

We can now determine κ by matching this equation to the Newtonian equation $\nabla^2\Phi = 4\pi G\rho$ in the appropriate limit. Consider a pseudo-cartesian coordinate system t, x, y, z where the spacetime is almost flat: $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $|h^{\mu\nu}| \ll 1$. We are going to keep terms only to leading order in $h^{\mu\nu}$. Consider a particle at rest in such a coordinate system. The geodesic equation for a particle at rest in such a coordinate system reduces to:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{tt}u^t u^t \tag{3.54}$$

since the particle's spatial four-velocity components are zero when the particle is at rest. We also note that (with the Latin letter i ranging only over the spatial indices x, y, z

$$\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right) = \frac{du^t}{d\tau}\frac{dx^i}{dt} + u^t\frac{d}{d\tau}\left(\frac{dx^i}{dt}\right) = 0 + u^t\frac{dt}{d\tau}\frac{d^2x^i}{dt^2} = u^t u^t\frac{d^2x^i}{dt^2} \tag{3.55}$$

since $dx^i/dt = 0$ if the particle is at rest. We see that the spatial acceleration of such a particle is therefore

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt} \tag{3.56}$$

This corresponds to the Newtonian equation $\vec{a} = -\vec{\nabla}\Phi$, where Φ is the Newtonian gravitational potential. Therefore, we identify Γ^i_{tt} as the thing that most closely corresponds to $\partial_i\Phi$ in the Newtonian limit.

$$\Gamma^i_{tt} = \partial_i\Phi \tag{3.57}$$

Now let's consider the time-time component of equation 3.53. Assume the fluid in the source is a perfect fluid but with $p_0 \ll \rho_0$, as would be the case for almost any Newtonian fluid. In our pseudo-cartesian coordinate system, we have

$$\begin{aligned} T^{tt} - \tfrac{1}{2}g^{tt}T + \Lambda g^{tt} &= \rho - \tfrac{1}{2}g^{tt}g_{\mu\nu}T^{\mu\nu} + \Lambda g^{tt} \\ &\approx \rho - \tfrac{1}{2}(-1)(-T^{tt} + T^{xx} + T^{yy} + T^{yy}) - \Lambda + \text{ order of } |h^{\mu\nu}| \\ &\approx \rho + \tfrac{1}{2}(-\rho) - \Lambda = \tfrac{1}{2}\rho - \Lambda \end{aligned} \tag{3.58}$$

neglecting the pressure terms. In the Newtonian limit, this is what appears on the right side of the time-time component of the Einstein equation.

On the left side, we have the time-time component of the Ricci tensor:

$$R^{tt} = g^{t\beta}g^{t\nu}R^{\alpha}{}_{\beta\alpha\nu} \tag{3.59}$$

But in our coordinate system, the metric is the same as $\eta^{\mu\nu}$ to leading order, so this becomes

$$R^{tt} \approx (-1)(-1)(R^t{}_{ttt} + R^x{}_{txt} + R^y{}_{tyt} + R^z{}_{tzt}) \tag{3.60}$$

$R^t{}_{ttt}$ is identically zero. Let's examine one of the other Riemann tensor components:

$$R^x{}_{txt} = \partial_x\Gamma^x_{tt} - \partial_t\Gamma^x_{xt} + \Gamma^x_{x\sigma}\Gamma^\sigma_{tt} - \Gamma^t_{x\sigma}\Gamma^\sigma_{xt} \tag{3.61}$$

The Christoffel systems also all involve derivatives of the metric, which will all be of order $|h^{\mu\nu}|$. The products of the Christoffel symbols will be therefore much smaller than the other derivative terms. Our Newtonian equation works for a static field, so we will also assume that the first term is zero. Therefore, we have $R^x{}_{txt} \approx \partial_x\Gamma^x_{tt}$, and the other components are similar. Therefore, we have

$$R^{tt} = \partial_x\Gamma^x_{tt} + \partial_y\Gamma^y_{tt} + \partial_z\Gamma^z_{tt} = \partial_x\partial_x\Phi + \partial_y\partial_y\Phi + \partial_z\partial_z\Phi = +\nabla^2\Phi \tag{3.62}$$

where I have use the identification in equation 3.57 The Einstein equation therefore reduces to

$$\nabla^2\Phi = \kappa\tfrac{1}{2}\rho - \Lambda \tag{3.63}$$

in the Newtonian limit. If this is to be equal to the Newtonian equation $\nabla^2\Phi = \kappa\frac{1}{2}\rho$ in that limit, we see that we must have $\kappa = 8\pi G$ and $\Lambda \ll 4\pi G\rho$.

We see from this last equation that the term involving Λ acts in the Newtonian limit as a *negative* energy density (assuming $\Lambda > 0$) that exists even in a vacuum. This would create a *repulsive* gravitational field. Since we do not observe this effect at the scale of the solar system, Λ must be small. But is it strictly zero?

When Einstein first proposed the theory of general relativity, the accepted model of the universe was a static model where stars and other cosmic objects are essentially at rest (except for random motions). Einstein added this term to his field equation when he developed his first cosmological model, because he found that without it, the Einstein equation predicted that the universe would either expand or contract but could not remain static. He found that a very small repulsive term in the field equation could cancel the gravitational attraction of stars and therefore allow the universe to be static, and a term of this form violates no other conditions on the field equation. He called Λ the "cosmological constant."

Not long after this, he determined that his static universe solution was not stable against small perturbations. Moreover, in 1929, observations by Hubble and others showed observationally that the universe was expanding, something that Einstein could have *predicted* if he had stuck with original field equation without the Λ. Einstein famously retracted the idea of a nonzero cosmological constant, calling it "the worst mistake I ever made." His subsequent work (and that of almost everyone else) assumed that $\Lambda = 0$.

However, providence may have played a role here. In 1998, astrophysicists discovered that the universe appears to be accelerating in its expansion, and a nonzero cosmological constant would provide the perfect explanation. Refined measurements of the cosmic microwave background have confirmed this result, strongly implying that $\Lambda/8\pi G \approx 0.7 \times 10^{-26}$ kg/m^3. This is small enough so that its effects cannot be observed within the solar system, but large enough to have significant effects on the evolution of the universe as a whole. Einstein's "worst mistake" has proved instead to be visionary with 80 years of hindsight.

Physicists current consider this term not to be part of the left side of the Einstein equation (the side that describes the curvature of spacetime) but rather an additional term added on the right, interpreting it as a stress-energy associated with the vacuum:

$$T^{\mu\nu}_{\text{vac}} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}. \tag{3.64}$$

It is important to remember, though, that the vacuum energy is so small that its affects are completely negligible unless we deal with scales larger than the largest galactic superclusters. Therefore whenever we are dealing with applications of the Einstein equation in situations other than cosmology, we will assume that $\Lambda = 0 \;\Rightarrow\; T^{\mu\nu}_{\text{vac}} = 0$.

Finally, then, we can write down the Einstein equation in its full and final glory:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = 8\pi G (T^{\mu\nu} + T^{\mu\nu}_{\text{vac}}) = 8\pi G T^{\mu\nu}_{\text{all}} \tag{3.65}$$

Where $G^{\mu\nu}$ is the **Einstein tensor**. The equivalent equation in a form that is easier to solve is

$$R^{\mu\nu} = 8\pi G (T^{\mu\nu}_{\text{all}} - \tfrac{1}{2} g^{\mu\nu} T_{\text{all}}) \tag{3.66}$$

Either equation represents 10 coupled, nonlinear differential equations in the 10 independent components of the metric. Even the "easy to solve" version will represent some challenges!

3.4.2 Exercise: The Vacuum Term on the Right of Equation 3.66.

Show that $T^{\mu\nu}_{\text{vac}} - \frac{1}{2} g^{\mu\nu} T_{\text{vac}} = +\Lambda g^{\mu\nu}/8\pi G$.

3.5 Is Local Energy Conservation Geometrically Necessary?

In the argument above, we derived the specific form of the Einstein Equation by requiring that it be consistent with the local law of conservation of energy and momentum $\nabla_\nu T^{\mu\nu} = 0$. But we can also view that conservation law as being *a logical consequence of* the Einstein equation rather than a condition imposed upon it. Consider how we might develop the Einstein equation $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ without this condition. $G^{\mu\nu}$ must still be a symmetric tensor constructed out of the Riemann tensor, and so still must involve at most $R^{\mu\nu}, g^{\mu\nu} R$, and $\Lambda g^{\mu\nu}$. However, a combination of these tensors with free coefficients would in general represent 10 (second-order, nonlinear) differential equations in the 10 independent components of the metric tensor $g_{\mu\nu}$, This should be enough information to *completely determine* those components.

But we *should not be able* to determine the components of $g_{\mu\nu}$ completely! The components of the metric tensor depend partly on our free and arbitrary choice of our coordinate system. Given four arbitrary coordinate transformation equations $x'^{\mu} = f^{\mu}(x^0, x^1, x^2, x^3)$, we should be able to transform the components $g_{\mu\nu}$ into $g'_{\mu\nu}$ and still have a completely valid solution to the Einstein equation in the same physical context. This means that the Einstein equation can place at most *six* equations-worth of constraint on the metric components $g_{\mu\nu}$, so that it takes four more arbitrary coordinate-choice equations to completely determine the metric's 10 independent components.

How can the 10 equations implied by $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ really only represent six equations-worth of constraint on the components $g_{\mu\nu}$? This can only be true if $G^{\mu\nu}$ is constructed so that it *automatically* satisfies four internal equations of constraint that make four of its ten equations linearly dependent on the other six. $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ turns out to be the *only* second-rank symmetric tensor we can construct out of the Riemann tensor and the metric that is (1) linear in second derivatives of the metric, (2) free of higher-order derivatives, (3) zero in flat spacetime, and (4) satisfies a four-equation internal constraint (in this case, $\nabla_\nu G^{\mu\nu} = 0$). [3]

But if $\nabla_\nu G^{\mu\nu} = 0$ is necessary to preserve the arbitrary nature of coordinates, then the Einstein equation *requires* that $\nabla_\nu T^{\mu\nu} = 0$. Therefore we see that local conservation of energy and momentum emerges *as a consequence* of our freedom to choose coordinates.

This is actually an example of Noether's theorem, which states quite generally that symmetries in the laws of physics imply conservation laws. In this case, we must construct the Einstein equation to be symmetric with regard to arbitrary transformations of the four coordinates: four conservation laws are the consequence.

The old view of the Einstein equation (Einstein's view!) assumes that the *source* is primary, and constraints that apply *a priori* to the source constrain the nature of the fields generated by that source. But the more modern approach (which also applies in quantum field theory) is that the *symmetry principles* (spacetime has no intrinsic geometry and can be described by completely arbitrary coordinates) are primary, and that the *field* expressing those principles places conservation-law constraints on the *source's* behavior.

It turns out that even the geodesic hypothesis is involved in this discussion! One can prove quite generally that the geodesic motion of objects emerges from $\nabla_\nu T^{\mu\nu} = 0$, which in turn follows from the coordinate-independence of the Einstein equation, as we have seen. In what follows, I will (for the sake of simplicity) only show that $\nabla_\nu T^{\mu\nu} = 0$ implies that particles of "dust" follow geodesics. Since a normal object (for example, a falling ball) consist of particles that for the most part move together (the pressure being relativistically negligible), this should be a good approximation for most realistic objects. (The general proof simply offers more mathematical difficulty with limited additional insight.)

Recall that the "dust" stress-energy tensor is $T^{\mu\nu} = \rho_0 u^\mu u^\nu$, where ρ_0 is the density of the dust in its rest LIF and u^μ is the common four-velocity of its particles at the event where we are evaluating the tensor. By the product rule,

$$0 = \nabla_\nu T^{\mu\nu} = \nabla_\nu(\rho_0 u^\mu u^\nu) = u^\mu \nabla_\nu(\rho_0 u^\nu) + \rho_0 u^\nu \nabla_\nu u^\mu \tag{3.67}$$

Now, we must have $-1 = g_{\alpha\mu}u^\alpha u^\mu$, which implies that

$$0 = \nabla_\nu(g_{\alpha\mu}u^\alpha u^\mu) = g_{\alpha\mu}u^\alpha \nabla_\nu u^\mu + g_{\alpha\mu}u^\mu \nabla_\nu u^\alpha = 2g_{\alpha\mu}u^\alpha \nabla_\nu u^\mu \tag{3.68}$$

where I have used the fact that $\nabla_\nu g_{\alpha\mu} = 0$ and done some creative renaming of bound indices in the last term. If one multiplies both sides of equation 3.67 by $g_{\alpha\mu}u^\mu$ and uses the equation above, one can show directly (see the exercise) that

$$\nabla_\nu(\rho_0 u^\nu) = 0 \tag{3.69}$$

If we substitute this back into equation 3.67, one finds that

$$\begin{aligned} 0 = u^\nu \nabla_\nu u^\mu = u^\nu \left(\frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\nu_{\beta\mu}u^\beta\right) &= \frac{\partial u^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} + \Gamma^\mu_{\beta\nu}u^\beta u^\nu \\ \Rightarrow \quad 0 = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\beta\nu}u^\beta u^\nu &= \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\beta\nu}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} \end{aligned} \tag{3.70}$$

(since $u^\mu \equiv dx^\mu/d\tau$. This is the geodesic equation. Therefore, we see that $\nabla_\nu T^{\mu\nu} = 0$ really does imply that an object consisting of particles that move together (with negligible internal pressure) must follow a geodesic.

So what, in the modern view, are the fundamental principles upon which general relativity is founded? One can argue that the principles are (1) coordinate systems are arbitrary, (2) gravity is geometry, and (3) the source of geometry is stress-energy and nothing else (no prior geometry). The first statement can be taken to be a generalized principle of relativity, but the I consider the second statement to be the true heart of general relativity, Einstein's truly original insight.

However, even though we now see that the geodesic hypothesis is secondary, I think that *pedagogically* it is the right place to begin, because it expresses the insight that gravity is geometry in an accessible way.

3.5.1 Exercise: Showing that $\nabla_\nu(\rho_0 u^\nu) = 0$.

Multiply both sides of equation 3.67 by $g_{\alpha\mu}u^\mu$ and use equation 3.68 and $-1 = g_{\alpha\mu}u^\alpha u^\mu$ to prove that $\nabla_\nu(\rho_0 u^\nu) = 0$.

3.6 Does a Gravitational Field Have Energy?

Note that our conservation law $\nabla_\nu T^{\mu\nu} = 0$ does not contain any terms that represent gravitational field energy. Electromagnetic fields and the other fields of quantum field theory definitely carry energy, and one can construct stress-energy tensors for such fields that one would include in the total stress-energy $\boldsymbol{T}$ when such fields are involved. But the Einstein equation should *build in* the stress-energy of the gravitational field somehow if such a concept exists, and it seems both that the field equation works (both theoretically and experimentally) without any terms in the stress-energy that would act either as the source of a a gravitational field or that would allow conversion of gravitational field energy into other forms of energy at the local level.

Let's think about how we might construct an expression for the energy density of a gravitational field. In electrodynamics, a static electric field has an energy density proportional to $|\vec{E}|^2$, where the field is defined such that $\vec{a} = (q/m)\vec{E}$. Now the equation for general relativity that corresponds to the latter equation is $d^2x^i/dt^2 = -\Gamma^i_{tt}$ (see equation 3.56). So we might expect the gravitational field energy density to be proportional to $(\Gamma^i_{tt})^2$. Perhaps we can construct a symmetric tensor where this is the time-time component? The problem with this idea would be that the Christoffel symbols are all zero in a LIF, so the energy density would have to be zero in LIF. Since a tensor quantity that is zero in one coordinate system is zero in all, we cannot construct a simple tensor quantity that expresses the energy density of a gravitational field this way.

This is actually emblematic of a deeper problem with energy conservation in general relativity. First note that my derivation of the "conservation" equation $\nabla_\nu T^{\mu\nu} = 0$ involved going to a LIF, where this equation becomes $\partial_\nu T^{\mu\nu} = 0$, and we saw that this says that energy and momentum flowing in and out of a tiny box centered at the origin was conserved. Now in flat spacetime, we can integrate $\partial_\nu T^{\mu\nu} = 0$ and use Gauss's law to link the flux of energy or momentum through an arbitrary closed surface to the energy or momentum enclosed. Therefore we can say that energy and momentum are globally conserved, meaning (for example) that energy coming in from a distant source can be absorbed locally and the loss at the distant source is exactly matched by the local gain.

But in an arbitrary curved spacetime, we *cannot* integrate $\nabla_\nu T^{\mu\nu} = 0$ over a large volume: Gauss's law does not apply in curved spacetime. Therefore we *cannot* make global statements about conservation of energy in general relativity at all. We can only make *local* statements that say that matter and energy behave locally in a LIF as if non-gravitational energy and momentum is conserved.

This should actually not be a surprise if we think about Noether's theorem. This theorem says that symmetries in the laws of physics lead to conservation laws, and in particular, that time-invariance in the laws of physics leads to energy conservation. But the Einstein equation does not have this such a symmetry: time is an intrinsic *part* of the equations, not a parameter. One would not therefore *expect* that a general global energy conservation law would apply in general relativity.

This does not mean that in specific cases we do not have symmetries that lead to conservation-like equations. We have seen that the fact that the Schwarzschild metric is independent of Schwarzschild coordinates t and ϕ lead to conserved quantities that look a lot like conservation of total energy (including a "gravitational potential energy" term) and conservation of angular momentum. Similarly, certain isotropies and symmetries in the metric for the universe imply that one can write the equation for the evolution of the universe's size that looks a lot like an energy conservation equation. But these are simply Noetherian consequences of particular symmetries involved in specific situations. None of these symmetries are "built into" general relativity.

One useful case where one can talk about energy conservation is in the situation where a region of curved spacetime is completely surrounded by an asymptotically flat spacetime. Then you can integrate over a closed surface in the flat spacetime, and say that energy crossing into or emerging from the enclosed region leads to a corresponding change in the energy inside (and even see changes in the gravitational field due to that energy), but one cannot say precisely *where* that energy is located inside the region.

When we consider gravitational waves in the last session, we will use a clever trick to calculate the "effective energy" that seems to be carried by a gravitational wave, energy that the source "loses." This is all meaningful only because we are surrounding the source with asymptotically flat spacetime (see the previous paragraph).

The bottom line is that conservation of energy and momentum in general relativity is a very slippery concept that must be approached with care and deliberation. I personally think that the argument from Noether's theorem is decisive: global energy conservation does not exist in general relativity. But others may think differently, and with some passion. (One of my colleagues witnessed a very heated argument (where voices were actually raised) over this very issue some years ago at a Caltech symposium.)

Homework Problems

3.1 The Christoffel symbols for longitude-latitude coordinates on the surface of a sphere (where $g_{\theta\theta} = R^2, g_{\phi\phi} = R^2 \sin^2\theta$, and $g_{\theta\phi} = g_{\phi\theta} = 0$ are $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$, and all others are zero. Show that this space is curved by evaluating an appropriate component of the Riemann tensor.

3.2 Argue that the stress-energy for an ideal gas of photons in a perfectly mirrored box at the origin of a LIF at rest with respect to the box is

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix} \tag{3.71}$$

3.3 Find the stress-energy tensor as a function of position inside a spherical ball of dust rotating around the z axis. (Assume that the ball of dust has only a weak gravitational field, so that the spacetime is essentially flat, and assume that the particles are essentially rotating as a solid sphere so that that $\vec{v} = \vec{r} \times \vec{\omega}$.)

3.4 Prove that $G^{\mu\nu} = 0$ if and only if $R^{\mu\nu} = 0$.

3.5 Calculate $T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T$ for a perfect fluid in an arbitrary coordinate system. Also find the components in terms of ρ_0 and p_0 of this quantity in a LOF where the fluid is at rest.

3.6 What are the components of the vacuum stress-energy in a LIF? What is unusual about these components? Will all observers in LIFs with various relative velocities agree on these components?

Notes

[1]Moore, *A General Relativity Workbook*, University Science Books, 2013, pp. 214-215.
[2]Misner, Thorne, and Wheeler, *Gravitation*, Freeman, 1973, p. 407.
[3]Misner, Thorne, and Wheeler, *Gravitation*, Freeman, 1973, p. 417.