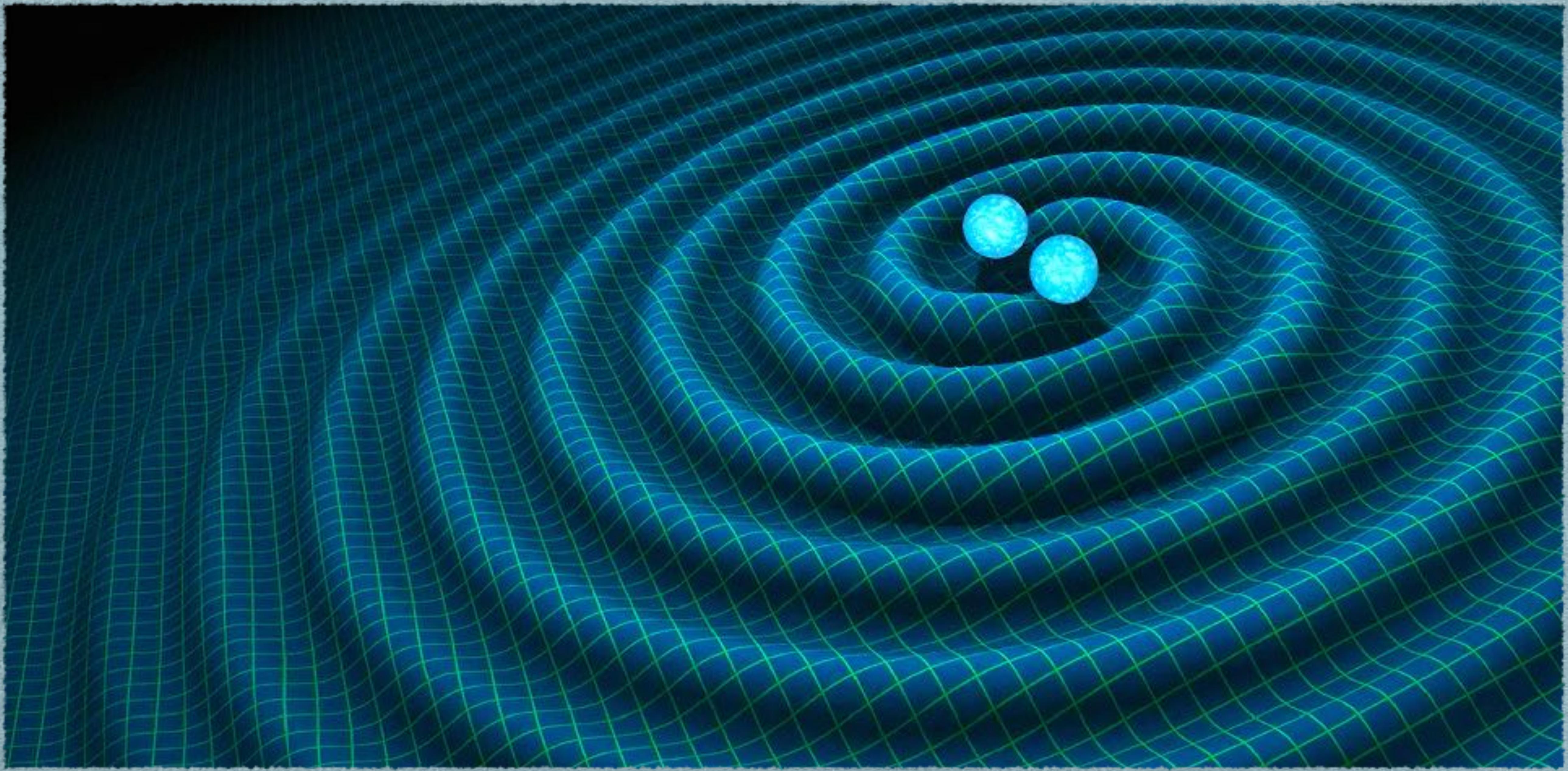

General Relativity and Gravitational Waves:

Session 3. The Einstein Equation

Thomas A. Moore — Les Houches — July 5, 2018

Overview of this session:

3.2 The Stress-Energy Tensor

3.3 The Riemann Tensor

3.4 Constructing the Einstein Equation

3.5 Is Local Energy Conservation Geometrically Necessary?

3.6 Does a Gravitational Field have Energy?

Our task

Find a tensor generalization of the Newtonian Poisson equation: $\nabla^2\phi = 4\pi G\rho\,.$

Two questions:

1. Is ρ mass density or energy density?
2. How does ρ transform?

Argument that ρ must be *energy:* thought experiment involving a box containing an electron-position pair.

How does energy density transform?

Dust is a fluid whose particles in a sufficiently small neighborhood all move with the same velocity.

Consider a tiny box containing N dust particles having volume V_0 in a LIF where the particles are at rest.
Number density is $n_0 = N/V_0$.

In a different LIF where the box is moving with speed v,
the box is Lorentz-contracted and so has volume $V = V_0\sqrt{1-v^2}$.
Number density is: $n = N/V = N/V_0\sqrt{1-v^2} = n_0/\sqrt{1-v^2}$.

How does energy density transform?

Note that in the second LIF, we can write:

$$u^\mu = \begin{bmatrix} 1/\sqrt{1-v^2} \\ v_x/\sqrt{1-v^2} \\ v_y/\sqrt{1-v^2} \\ v_z/\sqrt{1-v^2} \end{bmatrix} = \begin{bmatrix} u^t \\ v_x u^t \\ v_y u^t \\ v_z u^t \end{bmatrix}$$

So we can write:

$$n = \frac{N}{V} = \frac{N}{V_0\sqrt{1-v^2}} = n_0 u^t$$

$$\rho \equiv np^t = (n_0 u^t)(mu^t) = (n_0 m)u^t u^t = \rho_0 u^t u^t$$

How does energy density transform?

So ρ must be the time component of a second-rank tensor

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \quad \text{(for dust)}$$

Its other components:

$$\begin{aligned} T^{tx} &= \rho_0 u^t u^x = (n_0 m) u^t u^x \\ &= (n_0 u^t) m u^x = x\text{-momentum density} \end{aligned}$$

Alternatively:

$$= (n_0 u^t) m (u^t v_x) = n p^t v_x = \frac{(n A v_x\, dt) p^t}{A dt}$$

= energy flux in the x-direction

Perfect fluid stress-energy

At every location, divide up particles into dust classes, then add the dust stress-energies. If bulk fluid is at rest, most sums will cancel except terms like $\rho_0 u^x u^x = \rho_0 (u^t v_x)(u^t v_x) = \rho v_x^2$. So:

$$T^{\mu\nu} = \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{bmatrix} \quad \text{(for a perfect fluid at rest in a LIF)}$$

Tensor form: $T^{\mu\nu} = (\rho_0 + p_0) u^\mu u^\nu + p_0 \, g^{\mu\nu}$

$$T^{tt} = (\rho_0 + p_0) u^t u^t + p_0 \, \eta^{tt} = (\rho_0 + p_0) - p_0 = \rho_0$$
$$T^{xx} = (\rho_0 + p_0) u^x u^x + p_0 \, \eta^{xx} = 0 + p_0 = p_0 \quad \text{(and similarly for } T^{yy} \text{ and } T^{zz}\text{)}$$
$$T^{tx} = (\rho_0 + p_0) u^t u^x + p_0 \, \eta^{tx} = 0 + 0 \quad \text{(and similarly for other off-diagonal components)}$$

Conservation of energy and momentum

$\nabla_\nu T^{\mu\nu} = 0$ or at the origin of a LIF: $\partial_\nu T^{\mu\nu} = 0.$

Why? Consider energy flowing through a tiny box along *x* direction:

$$\left[(T^{tx})_{\text{left}} - T^{tx})_{\text{right}}\right] dy\, dz\, dt = \left(-\frac{\partial T^{tx}}{\partial x} dx\right) dy\, dz\, dt = -\frac{\partial T^{tx}}{\partial x} dx\, dy\, dz\, dt$$

Similar for *y*, *z* directions. Therefore net energy change from flow is

$$dE = \left[-\frac{\partial T^{tx}}{\partial x} - \frac{\partial T^{ty}}{\partial y} - \frac{\partial T^{tz}}{\partial z}\right] dx\, dy\, dz\, dt \quad = \frac{\partial T^{tt}}{\partial t} dt\, dx\, dy\, dz$$

if energy is conserved. Therefore:

$$0 = \left[\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{tx}}{\partial x} + \frac{\partial T^{ty}}{\partial y} + \frac{\partial T^{tz}}{\partial z}\right] dx\, dy\, dz\, dt \qquad \Rightarrow \qquad \partial_\nu T^{t\nu} = 0$$

Exercise: Fluid pressures

Pressure has units of force per unit area, which in units where we measure time and distance in meters, will be $(\mathrm{kg\,m/m^2})/\mathrm{m}^2 = \mathrm{kg/m^3}$ just like energy density $(\mathrm{kg\,m^2/m^2})/\mathrm{m}^3$. What random speeds would particles in a fluid need to have if the fluid pressure is even 1% of the fluid density in these units? What approximate temperature would this correspond to if the particles are electrons? (*Hints:* Remember that the average of v_x^2 will be $\frac{1}{3}$ times the average of v^2. The Newtonian approximation for the electrons' kinetic energy is adequate as a first approximation, and remember that $kT \approx \frac{1}{40}$ eV at 300 K and that an electron's rest energy is about 0.5 MeV.)

Which is a curved space?

$$g_{\mu\nu} = \begin{bmatrix} 1 + 4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix}, \qquad g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix}$$

We need a tensor quantity that somehow identifies a curved spacetime from an intrinsically "flat" spacetime. Candidate: something to do with

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\alpha = \nabla_\mu (\nabla_\nu A^\alpha) - \nabla_\nu (\nabla_\mu A^\alpha)$$

The Riemann Tensor

Note that

$$
\begin{aligned}
\nabla_\mu(\nabla_\nu A^\alpha) &= \partial_\mu(\nabla_\nu A^\alpha) - \Gamma^\beta_{\mu\nu}(\nabla_\beta A^\alpha) + \Gamma^\alpha_{\mu\sigma}(\nabla_\nu A^\sigma) \\
&= \partial_\mu(\partial_\nu A^\alpha + \Gamma^\alpha_{\nu\gamma}A^\gamma) - \Gamma^\beta_{\mu\nu}(\partial_\beta A^\alpha + \Gamma^\alpha_{\beta\delta}A^\delta) + \Gamma^\alpha_{\mu\sigma}(\partial_\nu A^\sigma + \Gamma^\sigma_{\nu\rho}A^\rho) \\
&= \partial_\mu\partial_\nu A^\alpha + (\partial_\mu\Gamma^\alpha_{\nu\gamma})A^\gamma + \Gamma^\alpha_{\nu\gamma}\partial_\mu A^\gamma - \Gamma^\beta_{\mu\nu}\partial_\beta A^\alpha - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\delta}A^\delta + \Gamma^\alpha_{\mu\sigma}\partial_\nu A^\sigma + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\rho}A^\rho
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_\mu(\nabla_\nu A^\alpha) - \nabla_\nu(\nabla_\mu A^\alpha) \quad = \\
&\quad + \cancel{\partial_\mu\partial_\nu A^\alpha} + (\partial_\mu\Gamma^\alpha_{\nu\gamma})A^\gamma + \cancel{\Gamma^\alpha_{\nu\gamma}\partial_\mu A^\gamma} - \cancel{\Gamma^\beta_{\mu\nu}\partial_\beta A^\alpha} - \cancel{\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\delta}A^\delta} + \cancel{\Gamma^\alpha_{\mu\sigma}\partial_\nu A^\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\rho}A^\rho \\
&\quad - \cancel{\partial_\nu\partial_\mu A^\alpha} - (\partial_\nu\Gamma^\alpha_{\mu\gamma})A^\gamma - \cancel{\Gamma^\alpha_{\mu\gamma}\partial_\nu A^\gamma} + \cancel{\Gamma^\beta_{\nu\mu}\partial_\beta A^\alpha} + \cancel{\Gamma^\beta_{\nu\mu}\Gamma^\alpha_{\beta\delta}A^\delta} - \cancel{\Gamma^\alpha_{\nu\sigma}\partial_\mu A^\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\rho}A^\rho \\
&\quad = (\partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta})A^\beta
\end{aligned}
$$

The quantity in parentheses is a tensor:

$$
R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta}
$$

The Riemann Tensor

Mnemonic: “3 to 4 is positive, twins bond inside”

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_\mu \mathit{\Gamma}^{\alpha}_{\nu\beta} - \partial_\nu \mathit{\Gamma}^{\alpha}_{\mu\beta} + \mathit{\Gamma}^{\alpha}_{\mu\sigma}\mathit{\Gamma}^{\sigma}_{\nu\beta} - \mathit{\Gamma}^{\alpha}_{\nu\sigma}\mathit{\Gamma}^{\sigma}_{\mu\beta}$$

3 4   3   4   4   3   3   4   4   3

Riemann Symmetries

Obvious from mnemonic: $R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}$

and with $R_{\alpha\beta\mu\nu} \equiv g_{\alpha\gamma}R^{\gamma}{}_{\beta\mu\nu}$, we also have

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

$$R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0 \quad \text{(Bianchi identity)}$$

The Riemann tensor evaluated at the origin of a LIF

At the origin of a LIF,

$$R_{\alpha\beta\mu\nu} = g_{\alpha\gamma} R^{\gamma}{}_{\beta\mu\nu} = g_{\alpha\gamma}(\partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta})$$
$$= g_{\alpha\gamma}\partial_\mu [\, \tfrac{1}{2} g^{\gamma\sigma}(\partial_\nu g_{\beta\sigma} + \partial_\beta g_{\sigma\nu} - \partial_\sigma g_{\nu\beta})\,]$$

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \tfrac{1}{2} g_{\alpha\gamma} g^{\gamma\sigma}(\cancel{\partial_\mu\partial_\nu g_{\beta\sigma}} + \partial_\mu\partial_\beta g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\nu\beta} \\ &\qquad - \cancel{\partial_\nu\partial_\mu g_{\beta\sigma}} - \partial_\nu\partial_\beta g_{\sigma\mu} + \partial_\nu\partial_\sigma g_{\mu\beta}) \\ &= \tfrac{1}{2}\delta^{\sigma}_{\alpha}(\partial_\mu\partial_\beta g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\nu\beta} - \partial_\nu\partial_\beta g_{\sigma\mu} + \partial_\nu\partial_\sigma g_{\mu\beta}) \\ &= \tfrac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\alpha\mu}) \end{aligned}$$

Mnemonic: "inner togetherness is positive"

$$R_{\alpha\beta\mu\nu} = \tfrac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\alpha\mu})$$

Example proof:

At the origin of a LIF,

$$
\begin{aligned}
R_{\beta\alpha\mu\nu} &= \tfrac{1}{2}(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\nu\partial_\alpha g_{\beta\mu}) \\
&= -\tfrac{1}{2}(\partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\alpha g_{\beta\nu} - \partial_\nu\partial_\beta g_{\mu\alpha}) \\
&= -\tfrac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} + \partial_\nu\partial_\alpha g_{\mu\beta} - \partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\beta g_{\alpha\mu}) \equiv -R_{\alpha\beta\mu\nu}
\end{aligned}
$$

Chart of possibly independent Riemann components

Because of $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ and $R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}$

$\alpha\beta \downarrow$ \ $\mu\nu \rightarrow$	01	02	03	12	13	23
01	$\underline{R_{0101}}$	R_{0102}	R_{0103}	R_{0112}	R_{0113}	R_{0123}
02	R_{0201}	$\underline{R_{0202}}$	R_{0203}	R_{0212}	R_{0213}	R_{0223}
03	R_{0301}	R_{0302}	$\underline{R_{0303}}$	R_{0312}	R_{0313}	R_{0323}
12	R_{1201}	R_{1202}	R_{1203}	$\underline{R_{1212}}$	R_{1213}	R_{1223}
13	R_{1301}	R_{1302}	R_{1303}	R_{1312}	$\underline{R_{1313}}$	R_{1323}
23	R_{2301}	R_{2302}	R_{2303}	R_{2312}	R_{2313}	$\underline{R_{2323}}$

But we also have $R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta}$

Also, $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$ which is usually trivially zero:

$R_{\alpha 1\mu 1} + R_{\alpha 11\mu} + R_{\alpha\mu 11} = 0$

Number left:

20!

Only nontrivial: $R_{0123} + R_{0312} + R_{0213} = 0$

Exercise:

In a two-dimensional space, only one Riemann tensor component is possibly independent: R_{0101}.

(a) Explain how and why all other components depend on this one (or are zero).

(b) We have seen that for the p, q parabolic coordinate system we have discussed in previous exercises, only one Christoffel symbol was nonzero $\Gamma^q_{pp} = 2c$, where c is a constant. Show that $R_{pqpq} = 0$ in this case, proving purely from the metric that parabolic coordinates must describe a flat space. For fast reference, the metric for parabolic coordinates is

$$g_{\mu\nu} = \begin{bmatrix} 1 + 4c^2p^2 & 2cp \\ 2cp & 1 \end{bmatrix} \tag{3.33}$$

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta}$$

Constructing the Einstein Equation

Looking for $G^{\mu\nu} = \kappa T^{\mu\nu}$ with $G^{\mu\nu} = G^{\nu\mu}$ and $\nabla_\nu G^{\mu\nu} = 0$

Candidate: the **Ricci** tensor $R_{\beta\nu} \equiv R^{\alpha}{}_{\beta\alpha\nu}$

With upper indices: $R^{\mu\nu} = g^{\mu\beta} g^{\nu\sigma} R^{\alpha}{}_{\beta\alpha\sigma} = g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} R_{\alpha\beta\gamma\sigma}$

It is symmetric: $$R^{\nu\mu} = g^{\nu\beta} g^{\mu\sigma} g^{\alpha\gamma} R_{\alpha\beta\gamma\sigma} = g^{\nu\beta} g^{\mu\sigma} g^{\alpha\gamma} R_{\gamma\sigma\alpha\beta}$$
$$= g^{\nu\sigma} g^{\mu\beta} g^{\gamma\alpha} R_{\alpha\beta\gamma\sigma} = g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} R_{\alpha\beta\gamma\sigma} \equiv R^{\mu\nu}$$

At the origin of a LIF:

$$\nabla_\nu R^{\mu\nu} = \nabla_\nu (g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} R_{\alpha\beta\gamma\sigma})$$
$$= \partial_\nu [\, g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} \tfrac{1}{2} (\partial_\beta \partial_\gamma g_{\alpha\sigma} + \partial_\alpha \partial_\sigma g_{\beta\gamma} - \partial_\beta \partial_\sigma g_{\alpha\gamma} - \partial_\alpha \partial_\gamma g_{\beta\sigma}) \,]$$
$$= g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} \tfrac{1}{2} (\partial_\nu \partial_\beta \partial_\gamma g_{\alpha\sigma} + \partial_\nu \partial_\alpha \partial_\sigma g_{\beta\gamma} - \partial_\nu \partial_\beta \partial_\sigma g_{\alpha\gamma} - \partial_\nu \partial_\alpha \partial_\gamma g_{\beta\sigma})$$
$$= g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} \tfrac{1}{2} (\partial_\nu \partial_\beta \partial_\gamma g_{\alpha\sigma} - \partial_\nu \partial_\beta \partial_\sigma g_{\alpha\gamma}) \neq 0 \quad \text{generally}$$

Constructing the Einstein Equation

What else might $G^{\mu\nu}$ contain? $g^{\mu\nu}$ and $g^{\mu\nu}R$

$$R \equiv g^{\beta\nu} R_{\beta\nu} = g^{\beta\nu} R^{\alpha}{}_{\beta\alpha\nu} = g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta\mu\nu}$$ **(curvature scalar)**

So let's see if we can construct something where

$$\nabla_\nu (R^{\mu\nu} + b g^{\mu\nu} R + \Lambda g^{\mu\nu}) = 0$$

The tensor gradient of the metric is zero:
at the origin of a LIF: $\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} = 0$

So our problem reduces to finding *b* such that

$$\nabla_\nu (R^{\mu\nu} + b g^{\mu\nu} R) = 0$$

Constructing the Einstein Equation

The key is the Bianchi Identity:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0$$

Note that:

$$g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} \nabla_\sigma R_{\alpha\beta\mu\nu} + g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} \nabla_\nu R_{\alpha\beta\sigma\mu} + g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} \nabla_\mu R_{\alpha\beta\nu\sigma} = 0$$

$$\nabla_\sigma (g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu}) + \nabla_\nu (g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\sigma\mu}) + \nabla_\mu (g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\nu\sigma}) = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - 2\nabla_\sigma R^{\gamma\sigma} = 0 \quad \Rightarrow \quad \nabla_\sigma (R^{\gamma\sigma} - \tfrac{1}{2} g^{\gamma\sigma} R) = 0$$

So we want to choose $b = -\frac{1}{2}$, making our field equation

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}$$

Constructing the Einstein Equation

An important side note:

$$g_{\mu\nu}R^{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\mu\nu}R + \Lambda g_{\mu\nu}g^{\mu\nu} = \kappa g_{\mu\nu}T^{\mu\nu}$$

$$R - 2R + 4\Lambda = -R + 4\Lambda = \kappa T$$

We can use this to show that we can write the Einstein equation in the following form:

$$R^{\mu\nu} = \kappa(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}T) + \Lambda g^{\mu\nu}$$

Exercise!

(a) Explain why $\nabla_\mu(g^{\gamma\sigma}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\nu\sigma}) = -\nabla_\sigma R^{\gamma\sigma}$. Specifically, which symmetries does one need, and how does one rename indices?

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \qquad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \qquad R_{\alpha\beta\mu\nu} = +R_{\mu\nu\alpha\beta}$$

$$R^{\mu\nu} \equiv g^{\mu\gamma}g^{\nu\sigma}g^{\alpha\beta}R_{\alpha\gamma\beta\sigma}$$

(b) Use the equations below to show $R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T) + \Lambda g^{\mu\nu}$

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}$$

$$-R + 4\Lambda = \kappa T$$

Determining κ

We can use the Newtonian limit to determine κ. We'll use pseudo-cartesian coordinates t, x, y, z in nearly flat spacetime:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \text{ where } |h^{\mu\nu}| \ll 1.$$

Note that $$\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right) = \frac{du^t}{d\tau}\frac{dx^i}{dt} + u^t\frac{d}{d\tau}\left(\frac{dx^i}{dt}\right)$$
$$= 0 + u^t\frac{dt}{d\tau}\frac{d^2x^i}{dt^2} = u^t u^t\frac{d^2x^i}{dt^2}$$

So the geodesic equation for a particle at rest: $\dfrac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{tt}u^t u^t$ becomes simply

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt}$$ which corresponds to $\vec{a} = -\vec{\nabla}\Phi$

So in the Newtonian limit, $\Gamma^i_{tt} = \partial_i \Phi$

Determining κ

Now look at the Einstein equation. For a fluid source where pressure is negligible compared to density, we have

$$\begin{aligned} T^{tt} - \tfrac{1}{2}g^{tt}T + \Lambda g^{tt} &= \rho - \tfrac{1}{2}g^{tt}g_{\mu\nu}T^{\mu\nu} + \Lambda g^{tt} \\ &\approx \rho - \tfrac{1}{2}(-1)(-T^{tt} + T^{xx} + T^{yy} + T^{yy}) - \Lambda \\ &\approx \rho + \tfrac{1}{2}(-\rho) - \Lambda = \tfrac{1}{2}\rho - \Lambda \end{aligned}$$

The left side becomes

$$R^{tt} = g^{t\beta}g^{t\nu}R^{\alpha}{}_{\beta\alpha\nu} \approx (-1)(-1)(R^{t}{}_{ttt} + R^{x}{}_{txt} + R^{y}{}_{tyt} + R^{z}{}_{tzt})$$

But $R^{x}{}_{txt} = \partial_x \Gamma^{x}_{tt} - \partial_t \Gamma^{x}_{xt} + \Gamma^{x}_{x\sigma}\Gamma^{\sigma}_{tt} - \Gamma^{t}_{x\sigma}\Gamma^{\sigma}_{xt} \approx \partial_x \Gamma^{x}_{tt}$

for a static field. The *ytyt* and *ztzt* components are similar, so

$$R^{tt} = \partial_x \Gamma^{x}_{tt} + \partial_y \Gamma^{y}_{tt} + \partial_z \Gamma^{z}_{tt} = \partial_x\partial_x \Phi + \partial_y\partial_y \Phi + \partial_z\partial_z \Phi = +\nabla^2 \Phi$$

Dark energy and the final Einstein equation(s)

We now consider the Λ term to be "vacuum energy" or "dark energy"

$$T^{\mu\nu}_{\mathrm{vac}} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}.$$

The final Einstein equations are thus

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = 8\pi G (T^{\mu\nu} + T^{\mu\nu}_{\mathrm{vac}}) = 8\pi G T^{\mu\nu}_{\mathrm{all}}$$

or equivalently:

$$R^{\mu\nu} = 8\pi G (T^{\mu\nu}_{\mathrm{all}} - \tfrac{1}{2} g^{\mu\nu} T_{\mathrm{all}})$$

Exercise: Show that $T^{\mu\nu}_{\mathrm{vac}} - \tfrac{1}{2} g^{\mu\nu} T_{\mathrm{vac}} = +\Lambda g^{\mu\nu}/8\pi G.$

Geometric Necessity

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G (T^{\mu\nu} + T^{\mu\nu}_{\text{vac}}) = 8\pi G T^{\mu\nu}_{\text{all}}$$

represents 10 (second-order, coupled, nonlinear differential) equations in 10 unknown metric components. We should be able to solve for $g_{\mu\nu}$.

But we *shouldn't* be able to solve! We still have freedom to choose coordinates! So coordinate freedom requires that $G^{\mu\nu}$ automatically satisfies four internal constraints so that we really have only six independent equations. We have ensured that by choosing *b* so that $\nabla_\mu G^{\mu\nu} = 0$. But this means that $\nabla_\mu T^{\mu\nu} = 0$.

So the symmetry of the Einstein equation under coordinate transformations *requires* local conservation of energy and momentum.

Geodesic equation from local energy/momentum conservation

Consider "dust" with $T^{\mu\nu} = \rho_0 u^\mu u^\nu$

$$0 = \nabla_\nu T^{\mu\nu} = \nabla_\nu(\rho_0 u^\mu u^\nu) = u^\mu \nabla_\nu(\rho_0 u^\nu) + \rho_0 u^\nu \nabla_\nu u^\mu$$

But we must have

$$0 = \nabla_\nu(g_{\alpha\mu} u^\alpha u^\mu) = g_{\alpha\mu} u^\alpha \nabla_\nu u^\mu + g_{\alpha\mu} u^\mu \nabla_\nu u^\alpha = 2 g_{\alpha\mu} u^\alpha \nabla_\nu u^\mu$$

One can multiply the first equation by $g_{\alpha\mu} u^\alpha$ and use the latter to show

$\nabla_\nu(\rho_0 u^\nu) = 0$ **Exercise:** Do this.

Substitute this back into the top equation to see that

$$0 = u^\nu \nabla_\nu u^\mu = u^\nu \left(\frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\nu_{\beta\mu} u^\beta \right) = \frac{\partial u^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} + \Gamma^\mu_{\beta\nu} u^\beta u^\nu$$

$$\Rightarrow \quad 0 = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\beta\nu} u^\beta u^\nu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\beta\nu} \frac{dx^\beta}{d\tau} \frac{dx^\nu}{d\tau}$$

Does the gravitational field have energy?

Problem: $\vec{a} \to -\Gamma^i_{tt}$ So $\rho_G \propto (\Gamma^i_{tt})^2$? But in a LIF $\Gamma^i_{tt} \to 0$

Problem: in flat spacetime we can integrate $\partial_\mu T^{\mu\nu} = 0$ to get global conservation laws. But we can't integrate $\nabla_\mu T^{\mu\nu} = 0$ in a general curved spacetime.

Problem: Noether's theorem says that because the laws of physics are independent of time, energy is conserved, and because they are independent of position, momentum is conserved. But what happens when the laws involve spacetime itself? Is $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ independent of time? What would that mean?

Specific cases where the idea works: (1) When the metric has symmetries. (2) When a region of curved spacetime is entirely surrounded by flat spacetime.