General Relativity and Gravitational Waves

Session 4: Solving the Einstein Equation

4.1 Overview of this Session

In the last session, we constructed the Einstein equation, which we saw is not just *a* core equation in general relativity but *the* core equation (even the geodesic hypothesis follows from it). However, solving the set coupled nonlinear second-order differential equations that it represents is no easy task. In this section, we will review methods that make solving the equation easier.

An overview of this session's sections follows:

4.2 The Schwarzschild Solution. First, I will guide you through the task of finding an *exact* solution of the Einstein equation using the Diagonal Metric Worksheet.

4.3 Interpreting the Schwarzschild Solution. This section will explore the crucial task of using the metric to explore what the Schwarzschild solution *means* physically. In particular, we will discuss how to set up a system that one might use to actually *measure* the Schwarzschild coordinates of events.

4.4 The Weak-Field Approximation. This section explores a different approach to solving the Einstein equation where we assume that the field is weak enough that we can ignore nonlinear terms in the equation. This approach will be crucial to us as we explore gravitational waves in our last session. We will find that in such a case, electrostatic analogies can help us solve the Einstein equation.

4.6 Gravitomagnetism. Indeed, in the weak-field limit, the analogy to electromagnetism is quite complete, as this section describes. We can use this analogy and our familiarity with electromagnetism to better understand the "magnetic" aspects of general relativity.

4.5 Gauge Freedom. In the weak-field approximation we often use pseudo-cartesian coordinates t, x, y, z, but our freedom to choose *exactly* how these coordinates are defined leads to freedom that (in analogy to electromagnetism) we call *gauge freedom.* This section explores exactly what freedoms we have.

4.2 The Schwarzschild Solution.

In many regards, solving the Einstein equation is no different than solving any complicated set of differential equations: (1) we use our physics and mathematical knowledge to make as good a *guess* for the solution as possible, and then (2) solve the equation for any unknown features of our trial solution. In this section, we will illustrate this process by deriving the Schwarzschild solution. Karl Schwarzschild derived this solution in his spare time in the trenches of World War I: [1] we ought to be able to do the same much more easily in this much more attractive setting.

The Guess. The Schwarzschild solution hopes to describe the gravitational field in the empty space surrounding a spherical and static object. We can take advantage of spherical symmetry and time-independence to make some choices about coordinates that will make solving the Einstein equation easier. First of all, First of all, we can imagine concentric spherical surfaces surrounding the star, which we will label by a monotonically increasing radial coordinate r that is constant on each surface. Each of these surfaces should (by the situation's spherical symmetry) *should* plausibly have the same geometry as the surface of a normal sphere in Euclidean space, so we will use ordinary spherical coordinates θ, ϕ to label points on those spheres, and assume that the ordinary spherical metric applies: $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ (meaning that $g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta,\ g_{\theta\phi} = g_{\phi\theta} = 0$). This portion of the metric also implicitly defines the meaning of the r coordinate: note that the circumference of an equatorial circle ($\theta = \text{constant} = \frac{1}{2}\pi, d\theta = 0$) is $\int ds = \int r\sin(\frac{1}{2}\pi)d\phi = r\int d\phi = 2\pi r$, so we see that we have defined the r coordinate of a given spherical surface to be the circumference of an equatorial circle on that surface divided by 2π.

Now, this part of the metric only applies to a single nested sphere: there is nothing in what we have done so far that precludes our giving each sphere its *own* set of θ and ϕ coordinates (by having the spheres' polar axes point in different directions). But we can require those coordinate systems to line up by requiring that lines of constant θ and constant ϕ be radial, that is, perpendicular to each spherical surface. This means that the r, θ, and ϕ basis vectors are mutually perpendicular, meaning that off-diagonal components involving r and either of the angular coordinates should be zero. So with these coordinate choices, the metric becomes

$$ds^2 = g_{rr}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.1}$$

Now, if we were to have a metric term like $g_{t\phi}\, dt\, d\phi$, then it would suggest that the geometry of spacetime treats positive displacements in ϕ differently than negative displacements, contradicting the idea of spherical symmetry. Therefore, we should be able to choose $g_{t\phi} = 0$ when we have spherical symmetry. A similar argument applies to $g_{t\theta}$. A nonzero value for g_{tr} would not violate spherical symmetry, but for a static source, we would expect to have *time-reversal* symmetry, which *would* be violated if this term were nonzero. So on the basis of these symmetries, we should be able define coordinates so that the metric has the form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.2}$$

before we even try to solve the Einstein equation. Moreover, spherical symmetry and time-independence would suggest that g_{tt} and g_{rr} will depend on the radial component r alone.

Now, this is an educated guess about what might successfully solve the Einstein equation in this situation, but it is still just a *guess*. If we find that this trial solution leads to a self-contradictory or trivial solution, then we can go back and try a more complicated guess.

Using the Diagonal Metric Worksheet. Since we are solving for the gravitational field in the empty space surrounding this object, the Einstein equation in this case (in its "easier-to-solve" form becomes simply $R^{\mu\nu} = 0$, because the stress-energy of empty space is essentially zero (we are treating the vacuum energy density as negligible here). So our next task is to evaluate the 10 independent components of the Ricci tensor for a metric of the form given above. The **Diagonal Metric Worksheet** (which is available online) lists all of the components of the Ricci tensor for a general diagonal metric of the form $ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2$, where dx^0, dx^1, dx^2, dx^3 are completely arbitrary coordinates and A, B, C, D are arbitrary functions of any or all of the coordinates. The worksheet uses a shorthand notation where

$$A_0 \equiv \frac{\partial A}{\partial x^0}, \quad B_{12} \equiv \frac{\partial^2 B}{\partial x^1 \partial x^2} \tag{4.3}$$

and so on. In our particular case, the coordinates $x^0 = t, x^1 = r, x^2 = \theta$, and $x^3 = \phi$, $C = r^2, D = r^2 \sin^2\theta$, and A and B are unknown functions of r alone. Our metric does not depend on t or ϕ, so any term involving a 0 or 3 subscript is zero. Also only D depends on θ, so any other terms involving a 2 subscript will be zero.

When I use the worksheet, I write above each term listed in the worksheet the equivalent term for the case in question as well as crossing out each term that is clearly zero. I then assemble all terms at the bottom. Figure 1 illustrates the process for calculating R_{tt} in this situation.

I will let you go through the process yourself as one of the homework problems. But when you do this, you will find that

$$R_{tt} = \frac{1}{2B}\left[\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right] \tag{4.4}$$

$$R_{rr} = \frac{1}{2A}\left[-\frac{d^2A}{dr^2} + \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2A}{Br}\frac{dB}{dr}\right] \tag{4.5}$$

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B}, \qquad R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{4.6}$$

and all of the off-diagonal components of $R_{\mu\nu}$ are identically zero. The Einstein equation in empty space requires that $R_{\mu\nu} = 0$. Assuming that $A \neq 0$ and $B \neq 0$, this means that

$$0 = 2BR_{tt} + 2AR_{rr} = \frac{2}{r}\frac{dA}{dr} + \frac{2A}{Br}\frac{dB}{dr} \tag{4.7}$$

(all terms but the last in each Ricci component cancel). Where $r \neq 0$, this becomes simply

$$0 = B\frac{dA}{dr} + A\frac{dB}{dr} = \frac{d}{dr}(AB) \quad \Rightarrow \quad AB = \text{constant} \tag{4.8}$$

Now, we would expect that as $r \to \infty$, the gravitational field would become negligible, meaning that the metric of spacetime should become flat spacetime, which implies that the metric should in this limit become

$$\frac{1}{2B}\frac{d^2A}{dr^2}$$

$$
\begin{aligned}
R_{00} = {} & 0 && + \frac{1}{2B}A_{11} && + \cancel{\frac{1}{2C}A_{22}} && + \cancel{\frac{1}{2D}A_{33}} \\
+ {} & 0 && - \frac{1}{2B}\cancel{B_{00}} && - \frac{1}{2C}\cancel{C_{00}} && - \frac{1}{2D}\cancel{D_{00}} \\
+ {} & 0 && + \frac{1}{4B^2}\cancel{B_0^2} && + \frac{1}{4C^2}\cancel{C_0^2} && + \frac{1}{4D^2}\cancel{D_0^2} \\
+ {} & 0 && + \frac{1}{4AB}\cancel{A_0}B_0 && + \frac{1}{4AC}\cancel{A_0}C_0 && + \frac{1}{4AD}\cancel{A_0}D_0
\end{aligned}
$$

$$-\frac{1}{4BA}\left(\frac{dA}{dr}\right)^2 \quad -\frac{1}{4B^2}\frac{dA}{dr}\frac{dB}{dr} \quad +\frac{1}{4Br^2}\frac{dA}{dr}(2r) \quad +\frac{1}{4Br^2\cancel{\sin^2\theta}}\frac{dA}{dr}2r\cancel{\sin^2\theta}$$

$$
\begin{aligned}
- {} & \frac{1}{4BA}A_1A_1 && - \frac{1}{4B^2}A_1B_1 && + \frac{1}{4BC}A_1C_1 && + \frac{1}{4BD}A_1D_1 \\
- {} & \frac{1}{4CA}\cancel{A_2}A_2 && + \frac{1}{4CB}\cancel{A_2}B_2 && - \frac{1}{4C^2}\cancel{A_2}C_2 && + \frac{1}{4CD}\cancel{A_2}D_2 \\
- {} & \frac{1}{4DA}\cancel{A_3}A_3 && + \frac{1}{4DB}\cancel{A_3}B_3 && + \frac{1}{4DC}\cancel{A_3}C_3 && - \frac{1}{4D^2}\cancel{A_3}D_3
\end{aligned}
$$

$$\boxed{R_{00} = \frac{1}{2B}\left[\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right]}$$

Figure 1: Using the Diagonal Metric Worksheet to work out the value of R_{tt} for the Schwarzschild trial metric.

$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ (the metric for flat spacetime in a spherical coordinate basis). We can enforce this by choosing the constant value of $AB = 1$, which is the value that AB must have at infinity if the metric is to be the flat-space metric.

If we now substitute $B = 1/A$ and equation 4.8 in the form

$$\frac{dB}{dr} = -\frac{B}{A}\frac{dA}{dr} \tag{4.9}$$

into equation 4.6, we can eliminate all references to B. With a bit of work, you can show that

$$0 = -r\frac{dA}{dr} + 1 - A \tag{4.10}$$

$$\Rightarrow \quad 1 = r\frac{dA}{dr} + A = \frac{d}{dr}(rA) \quad \Rightarrow \quad r = rA + K \quad \Rightarrow \quad A = 1 - \frac{K}{r} \tag{4.11}$$

where K is a constant of integration. Since $AB = 1$, we also know that

$$B = \frac{1}{A} = \frac{1}{1 - K/r} \tag{4.12}$$

We saw in the second session that if we choose $K = 2GM$, then a particle at rest accelerates at the rate that we would expect from Newtonian physics. With this identification, we have arrived at the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{4.13}$$

Since we seem to have a credible and self-consistent solution for the unknown terms in our trial metric, it seems that we made a good initial guess. We also see that the Diagonal Metric Worksheet makes solving the Einstein equation practical (imagine trying to work out all the Christoffel symbols and Ricci tensor components from first principles!).

4.2.1 Exercise: Filling in the Missing Steps.

Fill in the missing steps required to get from equations 4.6 and 4.9 to equation 4.10. Here is equation 4.6 again, for easy reference:

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B} \tag{4.14}$$

4.3 Interpreting the Schwarzschild Solution.

A theme that I am going to be repeating over and over is that coordinate names mean nothing: only the metric gives any kind of meaning to coordinates. In this section, I am going to illustrate some of the issues involved by discussing the meaning of the Schwarzschild coordinates as revealed by the metric equation.

We have already discussed the meaning of the r coordinate: it is a *circumferential* coordinate determined by measuring the circumference of a circle and dividing by 2π. One should be careful not to think of this coordinate as being a "radial" coordinate in the usual sense. For example, the radial distance between a point at coordinate r_0 and one at r_1 is not $r_1 - r_0$ but rather (calculating the arclength along a line where $dt = d\theta = d\phi = 0$

$$\Delta s = \int ds = \int_{r_0}^{r_1} \frac{dr}{1 - 2GM/r} = \left[r\sqrt{1 - \frac{2GM}{r}} + 2GM\tanh^{-1}\sqrt{1 - \frac{2GM}{r}} \right]_{r_0}^{r_1} \tag{4.15}$$

as long as $r > 2GM$. This begins to be approximately $r_1 - r_0$ when $r \gg 2GM$, but in general, $\Delta s > r_1 - r_0$. But we cannot even calculate this distance for $r < 2GM$ (for reasons we will discuss shortly). So in spite of its name, the Schwarzschild r coordinate is *not* a radial coordinate.

The Schwarzschild t coordinate is also not really a "time" coordinate. General clocks at rest in Schwarzschild spacetime do *not* measure t. The metric tells us that a clock at rest ($dr = d\theta = d\phi = 0$) in fact registers a proper time between events at its location of

$$\Delta\tau = \int \sqrt{-ds^2} = \int \sqrt{1 - \frac{2GM}{r}}\; dt = \left(1 - \frac{2GM}{r}\right)\Delta t \tag{4.16}$$

as long as $r > 2GM$. So a clock at rest only measures t in the limit that $r \to \infty$.

Light flashes must travel along worldlines such that $ds^2 = 0$. The metric tells us that for a radial light worldline (where $d\theta = d\phi = 0$), we will have

$$0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/R} \quad \Rightarrow \quad \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{r}\right)^2$$
$$\Rightarrow \quad \frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right) \tag{4.17}$$

as long as $r \geq 2GM$. The main thing that I want you to notice right now is that this equation implies that the coordinate t time required for a radially moving light flash to travel from one a given r-coordinate displacement is the same whether the flash is traveling inward or outward. This might seem to be obvious (as if it were required by physics), but this is actually a consequence of our definition of coordinates so that (among other things) the metric itself is independent of t. We *could* have defined coordinates differently, and we will in fact shortly see that our chosen definition of coordinates has certain problems.

But we now have enough of a picture of what the metric is saying to talk about how we could set up a machine for assigning Schwarzschild coordinates to events. By examining how must thrust a spacecraft needs to remain at rest, we can mark out locations in space that must have the same r-coordinate. Once we have marked out these locations, imagine constructing a lattice of girders that comprises a spherical surface around the object. We can lay out spherical coordinates θ, ϕ in the usual way on the surface of this sphere, and we can label each lattice intersections with the value of its θ and ϕ coordinate. We can also label each intersection with the (common) value of r that corresponds the value that we get when we measure the largest circumference of the sphere and divide by 2π.

Similarly, we set up other concentric spherical lattices, labeling their girder intersections similarly until we have a nice set going out to radii where $2GM/r$ is negligible. We lash all of these spheres together with radial girders perpendicular to the spheres' surfaces. Finally we arrange a set of strobe lights at various θ, ϕ coordinates on the outer sphere (at $r =$ "∞") connected to synchronized clocks at that so that they flash synchronously once every second (with an encoded message that specifies the clock time).

Finally, we set up a "t-meter" at every interior lattice intersection. The t-meter increments its value by 1 second every time it receives a flash from the strobe at infinity having its same angular position (so that the light flash is radial). These t-meters are not real clocks: they will in fact tick faster by an r-dependent amount than a real clock placed at the same intersection.

Finally, we need to synchronize all the t-meters. Since the radial light travel Δt going in is the same as going out from, a worker at each intersection can send a light flash out to its corresponding clock at "∞" and receive a response reflected from it. The Δt it takes a signal to get from the clock at "∞" to the t-meter in question is half the difference between the arrival and departure t-values as registered by the local t-meter. The worker then knows that the local t-value is Δt larger than the time encoded in each flash, and so set the t-meter accordingly.

Now, whenever an event occurs in the lattice, an observer can record the labels for r, θ, ϕ at the nearest intersection and record the value displayed on the t-meter at that intersection at the time the event occurred. This will allow us to assign coordinates to each event.

However, this all falls apart as we try to extend the lattice down past $r = 2GM$. We have already seen that we cannot determine radial distances past this radius. A clock at rest at this radius registers no time, so a worker at this location trying to set the t-meter will see all the flashes coming at once. Indeed a clock or worker at rest at this location would have to be constructed out of photons, since being at rest at this location corresponds to traveling along a light worldline.

Indeed, the metric for $r < 2GM$ implies that $g_{tt} > 0$ and $g_{rr} < 0$. In a diagonal metric, the negative metric component indicates the time-like coordinate, so in spite of its name, the r coordinate is the time-like coordinate and t is just another spatial component. But the metric says that r still describes the circumferences of imaginary spherical surfaces at constant t and r, which therefore become smaller and smaller as r becomes smaller. And for someone that falls in past the $r = 2GM$ sphere, r must go inward with the same inevitability as time goes forward for us, and $r = 0$ is this person's future in a very literal sense. Even light must move inward (toward the future) inside this radius.

The point is that we cannot extend our spherical lattice scheme to $r = 2GM$ or inward. In fact, one can show that the vertical force one would have to exert on an object (according to a local observer) to hold it at rest goes to infinity as $r \to 2GM$, so any girder that we try to lower to this radius would get torn apart. We cannot set up t-meters at rest, and the light signals that we need to use to set up and synchronize those t-meters end up going inward, not outward.

Of course, $2GM$ for an object the mass of the sun is about 3 km. The surface of most physical objects is well outside this radius, and the Schwarzschild solution (because it is a vacuum solution), does not apply *inside* such an object. So all of this weird behavior does not even apply to most objects. But the metric tells us that such problems will arise if an object's mass were ever to be compressed inside its **Schwarzschild radius** $r = 2GM$ (so that the vacuum solution applies down to this radius), then the mass can never come out, but rather must end up at the endpoint of its future, which is $r = 0$: the object becomes a **black hole**.

Others are going to talk to you more about black holes. My purpose here has been entirely to give you practice in looking beyond the *names* of the coordinates to what the metric says those coordinates actually *mean*. This will continue to be an issue for us as we move forward.

4.3.1 Exercise: Kruskal-Szekeres Coordinates.

The Kruskal-Szekeres coordinate system is an alternative solution to the empty-space Einstein equation in the spherically symmetric case. The Kruskal-Szekeres metric is

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM}(dv^2 - du^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.18}$$

where r is not a coordinate here but is a shorthand for a function $r(v, u)$ implicitly defined by

$$\left(\frac{r}{2GM} - 1\right) e^{r/2GM} = u^2 - v^2 \tag{4.19}$$

Note that the components of this metric do *not* behave badly at $r = 2GM$ (which corresponds to events where $u^2 = v^2$).

(a) What is the time coordinate in this metric? Is it a time coordinate for all values of r?

(b) What kind of worldline does a particle at fixed r follow in u, v coordinates? Is r fixed if u is fixed? If v is fixed?

(c) Argue that the value of the function r still corresponds to the circumference of an equatorial ($\theta = \pi/2$) circle divided by 2π but now evaluated for fixed u and v (instead of fixed r and t as in Schwarzschild coordinates). Is r fixed if u *and* v are fixed?

4.4 The Weak-Field Approximation.

Another approach to solving the Einstein equation is to take what is known as the **weak-field limit**. "Weak" fields in this context are actually not all that weak by astrophysical standards. One would actually need to be close to a black hole or a neutron star for the weak-field approximation to break down. No gravitational field in the solar system would even come close to violating this approximation. This is also the path to understanding both the generation and detection of gravitational waves.

We describe a gravitational field as "weak" in a region of spacetime if we can describe the spacetime using almost-cartesian coordinates t, x, y, z whose metric is such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \quad h_{\mu\nu} = h_{\nu\mu} \text{ and } |h_{\mu\nu}| \ll 1 \tag{4.20}$$

where $\eta_{\mu\nu}$ is the flat-spacetime metric. Not that the "almost-cartesian" coordinates here are not the same as true cartesian coordinates (for which the metric would be exactly $\eta_{\mu\nu}$).

Taking the "weak-field limit" means that we will drop terms of order $|h_{\mu\nu}|^2$ and higher in all the equations that follow. To this level of approximation, the inverse metric $g^{\mu\nu}$ turns out to be

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{where } h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} \tag{4.21}$$

To see this, note that

$$\begin{aligned} g^{\mu\nu}g_{\nu\sigma} &= (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\sigma} + h^{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} - h^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + h^{\mu\nu}h_{\nu\sigma} \\ &= \delta^\mu_\sigma - \eta_{\nu\sigma}(\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}) + \eta^{\mu\nu}h_{\nu\sigma} + (\text{dropped}) \\ &= \delta^\mu_\sigma - \eta^{\mu\alpha}(\eta_{\sigma\nu}\eta^{\nu\beta})h_{\alpha\beta}) + \eta^{\mu\nu}h_{\nu\sigma} \\ &= \delta^\mu_\sigma - \eta^{\mu\alpha}\delta^\beta_\sigma h_{\alpha\beta} + \eta^{\mu\nu}h_{\nu\sigma} = \delta^\mu_\sigma - \cancel{\eta^{\mu\alpha}h_{\alpha\sigma}} + \cancel{\eta^{\mu\nu}h_{\nu\sigma}} = \delta^\mu_\sigma \end{aligned} \tag{4.22}$$

implying that the stated inverse does satisfy its definition equation to the order in question.

This means that we can raise or lower the indices of any quantity that is of order of $h_{\mu\nu}$ using the *flat-space* inverse metric $\eta^{\mu\nu}$ or metric $\eta_{\mu\nu}$, respectively. For example

$$h^{\mu}{}_{\nu} \equiv g^{\mu\alpha} h_{\alpha\nu} \approx (\eta^{\mu\alpha} - h^{\mu\alpha}) h_{\alpha\nu} = \eta^{\mu\alpha} h_{\alpha\nu} + (\text{dropped}) \tag{4.23}$$

since the dropped term is of order $|h_{\alpha\nu}|^2$. This will be very important to keep in mind in what follows.

In particular, note that Christoffel symbols in this limit become

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \approx \tfrac{1}{2}\eta^{\alpha\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) \tag{4.24}$$

because the $\eta_{\mu\nu}$ matrix is a constant, and we can throw away the correction to the inverse metric because it will only lead to terms of order $|h_{\alpha\nu}|^2$. Similarly, the expression for the Riemann tensor in this limit is

$$R_{\alpha\beta\mu\nu} = \tfrac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}) \tag{4.25}$$

The derivation in this case is essentially the same as the derivation we did last time for the Riemann tensor in a LIF. At the origin of a LIF, the Christoffel symbols are exactly zero, so their products vanish, leaving only terms involving their derivatives. Also the metric in a LIF is exactly $\eta_{\mu\nu}$, so we raise or lower indices with that matrix. In the weak field limit, we throw away the products of the Christoffel symbols not because they are zero but because they are of order $|h_{\mu\nu}|^2$. Also we raise and lower indices with $\eta_{\mu\nu}$ not because the metric is $\eta_{\mu\nu}$ but because including the correction term will only lead to terms of order $|h_{\mu\nu}|^2$, Also only the $h_{\mu\nu}$ part of $g_{\mu\nu}$ varies, so only hs appear in the derivatives. Since the math is basically the same, the indices still obey the "inner togetherness is positive" mantra that we saw before.

The Einstein equation (in arbitrary coordinates, but with both indices lowered) says that

$$R_{\beta\nu} = 8\pi G(T_{\beta\nu} - \tfrac{1}{2} g_{\beta\nu} T) \quad \text{where } T \equiv g_{\mu\nu} T^{\mu\nu} \tag{4.26}$$

Using the definition of the Ricci tensor and the expression for the Riemann tensor given above, we see that

$$R_{\beta\nu} \equiv g^{\alpha\mu} R_{\alpha\beta\mu\nu} \approx \tfrac{1}{2}\eta^{\alpha\mu}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}) \tag{4.27}$$

So this is what appears on the left side of the Einstein equation. Though this makes the differential equations we have to solve linear, the result is still not very pretty.

Though it will not look at first like it is going to help, it actually does help to define the **trace-reversed metric perturbation**

$$H_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h \quad \text{where } h \equiv \eta^{\alpha\beta} h_{\alpha\beta} \tag{4.28}$$

This is called "trace-reversed" because

$$H \equiv \eta^{\mu\nu} H_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu} h = h - \tfrac{1}{2}\delta^{\mu}_{\mu} h = h - 2h = -h \tag{4.29}$$

This also means that

$$h_{\mu\nu} = H_{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu} h = H_{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} H \tag{4.30}$$

If we substitute this into equation 4.27, we find that

$$\begin{aligned} R_{\beta\nu} &\approx \tfrac{1}{2}(\partial_\beta\partial_\mu \eta^{\alpha\mu}[H_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu} H] + \partial_\alpha\partial_\nu \eta^{\alpha\mu}[H_{\beta\mu} - \tfrac{1}{2}\eta_{\beta\mu} H] - \eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\beta\partial_\nu H) \\ &= \tfrac{1}{2}(\partial_\beta\partial_\mu[H^{\mu}{}_{\nu} - \cancel{\tfrac{1}{2}\delta^{\mu}_{\nu} H}] + \partial_\alpha\partial_\nu[H^{\alpha}{}_{\beta} - \cancel{\tfrac{1}{2}\delta^{\alpha}_{\beta} H}] - \partial^\mu\partial_\mu h_{\beta\nu} + \cancel{\partial_\beta\partial_\nu H}) \\ &= \tfrac{1}{2}(\partial_\beta\partial_\mu[\eta_{\nu\sigma} H^{\mu\sigma}] + \partial_\alpha\partial_\nu[\eta_{\beta\sigma} H^{\alpha\sigma}] - \Box^2 h_{\beta\nu}) \end{aligned} \tag{4.31}$$

and the Einstein equation becomes

$$\Box^2 h_{\beta\nu} - \eta_{\nu\sigma}\partial_\beta\partial_\mu H^{\mu\sigma} - \eta_{\beta\sigma}\partial_\alpha\partial_\nu H^{\alpha\sigma} = -16\pi G(T_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu} T) \tag{4.32}$$

where $\Box^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial^2/\partial_t^2 + \nabla^2$. (Note that because this equation makes it clear that $T_{\mu\nu}$ is of the same order as $h_{\mu\nu}$, there is no point distinguishing between $g_{\mu\nu}$ and $\eta_{\mu\nu}$ on the right side.) Now, this might not seem much of an improvement, but as we will see below, we can always find a coordinate transformation that sets $\partial_\mu H_{\mu\nu} = 0$, and this sets both of the H-terms in the equation above to zero. So solving the Einstein equation is (in this limit) solving the coupled pair of equations

$$\Box^2 h^{\beta\nu} = -16\pi G(T^{\beta\nu} - \tfrac{1}{2}\eta^{\beta\nu}T) \quad \text{and} \quad 0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h) \tag{4.33}$$

where I have raised the β, ν indices on both sides.

Now this is actually not so bad. We can compare this equation to the analogous time-dependent equation for the electric potential, where

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = -\frac{\rho_c}{\varepsilon_0} = -4\pi k\rho_c \tag{4.34}$$

where ρ_c is the charge density and $k = 1/4\pi\varepsilon_0$ is the Coulomb constant (which is directly analogous to the gravitational constant G). You may remember that in *static* situations, we can calculate the potential $\phi(t, \vec{R})$ at position $\vec{R}$ is do divide the source into tiny volume elements, treat each volume element as a point particle with charge $q = \rho_c dV$, use the formula for the potential for a point charge $\phi = kq/s$ (where $s \equiv |\vec{R} - \vec{r}|$ is the distance between the field point at position $\vec{R}$ and the volume element's position $\vec{r}$), and then sum over all volume elements. Such a potential will satisfy the time-*independent* version of equation 4.34. You may also know that the solution to the time-*dependent* equation is the same thing except we account for the light travel time that it takes to get from the source-element's position $\vec{r}$ and the field point position $\vec{R}$. By analogy, then, the solution to the weak-field Einstein equation (equation 4.33) will be, quite generally,

$$h^{\mu\nu}(t, \vec{R}) = 4G\int_{\text{src}} \frac{\overline{T}^{\mu\nu}(t - s, \vec{r})\, dV}{s} \quad \text{where} \quad \overline{T}^{\mu\nu} \equiv T^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}T \tag{4.35}$$

subject to the condition that $\partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h) = 0$.

4.4.1 Exercise: A Static Spherical Star.

Consider a static spherical star with total mass M. Assume that pressure is negligible, so that its only nonzero stress-energy component is $T^{tt} = \rho$, where ρ is a function of radius alone.

(a) Show that $\overline{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T$ has components $\overline{T}^{tt} = \overline{T}^{xx} = \overline{T}^{yy} = \overline{T}^{zz} = \frac{1}{2}\rho$ at every point.

(b) We know from electrostatics that the potential outside a spherical source is the same as that for particle with same charge located at the source's center. Using that analogy, find all components of $h^{\mu\nu}$ at a point outside the star that is a distance r from the star's center.

(c) What is $H^{\mu\nu}$ for this solution? Show that $\partial_\mu H^{\mu\nu} = 0$.

(d) What is g_{tt} outside the star? Does this look familiar? (Don't forget to lower the indices on $h^{\mu\nu}$.)

4.5 Gravitomagnetism.

The electromagnetic analogy actually extends farther than one might imagine. Suppose we define a **gravitoelectric** potential Φ_G and a **gravitovector potential** $\vec{A}_G$ such that

$$\Phi_G \equiv -\tfrac{1}{8}(h^{tt} + h^{xx} + h^{yy} + h^{zz}) \quad \text{and} \quad A_G^i \equiv -\tfrac{1}{4}h^{ti} \tag{4.36}$$

Here (and throughout this section) I will use an established convention that Latin letters range only over *spatial* indices. Now, the time component of our coordinate condition $\partial_\mu H^{\mu\nu} = 0$ tells us that

$$\begin{aligned}
0 &= \partial_\mu(h^{\mu t} - \tfrac{1}{2}\eta^{\mu t}h) = \partial_t h^{tt} + \partial_i h^{it} - \tfrac{1}{2}(-1)\partial_t(-h^{tt} + h^{xx} + h^{yy} + h^{zz}) \\
&= \partial_i h^{it} + \tfrac{1}{2}\partial_t(h^{tt} + h^{xx} + h^{yy} + h^{zz}) = -4\vec{\nabla}\cdot\vec{A}_G - 4\partial_t\Phi_G \\
&\Rightarrow \quad \vec{\nabla}\cdot\vec{A}_G = -\frac{\partial\Phi_G}{\partial t}
\end{aligned} \tag{4.37}$$

as is true for the analogous electromagnetic potentials in the Lorenz gauge. If we now define a **gravitoelectric field** such that $\vec{E}_G \equiv -\vec{\nabla}\Phi_G - \partial\vec{A}_G/\partial t$, we find that

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E}_G &= \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi_G - \frac{\partial\vec{A}_G}{\partial t}\right) = -\nabla^2\Phi_G - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}_G\right) = -\Box^2\Phi_G \\
&= +\tfrac{1}{8}\Box^2(h^{tt} + h^{xx} + h^{yy} + h^{zz})
\end{aligned} \tag{4.38}$$

But by the weak-field Einstein equation, this is

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E}_G &= -2\pi G(T^{tt}+T^{xx}+T^{yy}+T^{zz}+\tfrac{1}{2}T-\tfrac{1}{2}T-\tfrac{1}{2}T-\tfrac{1}{2}T)\\
&= -2\pi G(T^{tt}+T^{xx}+T^{yy}+T^{zz}-[-T^{tt}+T^{xx}+T^{yy}+T^{zz}])\\
&= -4\pi G T^{tt} = -4\pi G\rho \qquad (4.39)
\end{aligned}
$$

If we define the **gravitomagnetic field** $\vec{B}_G = \vec{\nabla}\times\vec{A}_G$, then we have

$$\vec{\nabla}\times\vec{B}_G = \vec{\nabla}\times\vec{\nabla}\times\vec{A}_G = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}_G) - \nabla^2 A_G = \vec{\nabla}\left(-\frac{\partial\Phi_G}{\partial t}\right) - \nabla^2 A_G \qquad (4.40)$$

where I have used a well-known vector identity and the Lorenz condition (equation 4.37). If I now add and subtract a $\partial^2\vec{A}_G/\partial t^2$ term in the middle of this, we get

$$
\begin{aligned}
\vec{\nabla}\times\vec{B}_G &= \frac{\partial}{\partial t}\left(\vec{\nabla}\Phi_G - \frac{\partial\vec{A}_G}{\partial t}\right) + \frac{\partial^2\vec{A}_G}{\partial t^2} - \nabla^2 A_G = \frac{\partial\vec{E}_G}{\partial t} - \Box^2\vec{A}_G\\
&= \frac{\partial\vec{E}_G}{\partial t} + \tfrac{1}{4}\Box^2 h^{ti} = \frac{\partial\vec{E}_G}{\partial t} - 4\pi G(T^{ti} - \tfrac{1}{2}\eta_{ti}T) = \frac{\partial\vec{E}_G}{\partial t} - 4\pi G(\vec{J}+0) \qquad (4.41)
\end{aligned}
$$

where $J^i \equiv T^{ti}$ = energy flux in the i direction. You can easily use vector identities to derive the analogues to Gauss's law for the magnetic field and Faraday's law. So in this weak-field limit, we have a complete set of gravitomagnetic Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E}_G = -4\pi G\rho \qquad (4.42)$$

$$\vec{\nabla}\times\vec{B}_G - \frac{\partial\vec{E}_G}{\partial t} = -4\pi G\vec{J} \qquad (4.43)$$

$$\vec{\nabla}\cdot\vec{B}_G = 0 \qquad (4.44)$$

$$\vec{\nabla}\times\vec{E}_G + \frac{\partial\vec{B}_G}{\partial t} = 0 \qquad (4.45)$$

Note the minus signs in the first two equations compared to the electromagnetic equations. The minus sign in "Gauss's law" is because the gravitational field created by a positive mass density is attractive, not repulsive. The minus sign in the "Ampere-Maxwell law" implies that you should use a *left*-hand rule for predicting the direction of $\vec{B}_G$ created by a current where one would use a right-hand rule for a normal magnetic field $\vec{B}$.

These equations apply very generally in the weak-field limit. However, they do not mean much unless these fields are connected to the motion of particles in electromagnetic-like ways. Through a complicated calculation, one can show that under the right conditions, the Newtonian acceleration $\vec{a} \equiv d^2\vec{x}/dt^2$ of a particle moving with velocity $\vec{v}$ in this weak field is

$$\vec{a} \approx \vec{E}_G + \vec{v}\times 4\vec{B}_G \qquad (4.46)$$

This equation (except for the factor of 4) looks a lot like the electromagnetic Lorentz force law! However, the equation only applies when (1) the field is essentially static, (2) the fluid pressure in the field's source is negligible, (3) the bulk fluid velocity in the source is small enough that $u^i u^j$ is negligibly small compared to 1, and (4) the particle responding to the field is moving slowly enough that its v^2 is negligible compared to 1. If *any* of these constraints are violated, this gravitomagnetic force law gets more terms that do not correspond to anything in electrodynamics.

Fortunately, however, these conditions apply in many circumstances of astrophysical interest. This is *huge*, because it means that you can apply whatever you know about electromagnetic fields to gravitomagnetic fields (as long as you remember the left-hand rule for sources of $\vec{B}_G$ and the four-fold amplification of the effect of $\vec{B}_G$). Even in cases where the conditions do not strictly apply, the electromagnetic analogy can help you understand the qualitative consequences of being near spinning objects or significant energy fluxes.

In particular, note that a spinning object will create a gravitomagnetic field that causes a moving particle to experience an acceleration component proportional to and perpendicular to its velocity. An object that is initially falling radially toward a spinning star in its equatorial plane will be deflected by gravitomagnetic effects in the direction of the star's spin (an effect misleadingly called "frame-dragging".) Spinning objects will precess in a gravitomagnetic field (an effect that was measured recently by Gravity Probe B.[2]) Spinning stars in a close binary system undergo electromagnetic-like spin-spin and spin-orbit interactions that cause the stars and the system's orbital plane to precess in ways that modulate the gravitational waves emitted by that system (something I and my research students are currently studying).

In short, this analogy is a powerful tool for developing one's insight about how systems behave in general relativity, even beyond the weak-field limit.

4.5.1 Exercise: Basic Gravitomagnetism.

(a) Use the gravitomagnetic analogy to argue that a particle falling initially radially in the equatorial plane of a spinning object does indeed experience an acceleration component that deflects in the direction of the object's spin. (Don't forget the left-hand rule!)

(b) Imagine a particle in a circular orbit in the equatorial plane of a spinning object. Suppose the particle is orbiting in the same direction as the object is rotating. Will the period of the particle's orbit be affected by the object's spin? If so, will it be longer or shorter than if the object were not spinning?

(c) Imagine an uncharged particle of dust is moving initially parallel to a relativistic stream of particles (perhaps a jet from a quasar). Will gravitomagnetic effects repel or attract it to the stream? Are those effects likely to be as large as the basic gravitoelectric attraction it experiences toward the stream?

4.6 Gauge Freedom.

We simplified the weak-field Einstein equation by requiring that that $0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)$. What gives us the freedom to do this?

We are working in "nearly cartesian" coordinates t, x, y, z for which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $|h_{\mu\nu}| \ll 1$. The latter restriction is actually not very restrictive: there are still infinitely many "nearly cartesian" coordinate systems having different $h_{\mu\nu}$s that still have $|h_{\mu\nu}| \ll 1$. We can take advantage of this microscale coordinate freedom to pick the particular coordinate systems that make solving the Einstein equation easier.

To see just how much freedom we have, consider making a small adjustment to a coordinate system that is already "nearly cartesian:"

$$x'^{\,\alpha} = x^\alpha + \xi^\alpha \quad \text{where} \;\; \xi^\alpha = \xi^\alpha(t,x,y,z) \text{ and } \; |\xi^\alpha| \ll 1 \tag{4.47}$$

Note that this means that the inverse transformation is $x^\alpha = x'^{\,\alpha} - \xi^\alpha$. The transformation partials that we need to transform the metric in this case are

$$\begin{aligned}\frac{\partial x^\beta}{\partial x'^{\,\alpha}} &= \frac{\partial}{\partial x'^{\,\alpha}}\left(x'^{\,\beta} - \xi^\beta\right) = \delta^\beta_\alpha - \frac{\partial \xi^\beta}{\partial x'^{\,\alpha}} = \delta^\beta_\alpha - \frac{\partial x^\sigma}{\partial x'^{\,\alpha}}\frac{\partial \xi^\beta}{\partial x^\sigma}\\ &= \delta^\beta_\alpha - \left(\delta^\sigma_\alpha - \frac{\partial \xi^\sigma}{\partial x'^{\,\alpha}}\right)\frac{\partial \xi^\beta}{\partial x^\sigma} \approx \delta^\beta_\alpha - \delta^\sigma_\alpha\frac{\partial \xi^\beta}{\partial x^\sigma} = \delta^\beta_\alpha - \partial_\alpha \xi^\beta\end{aligned} \tag{4.48}$$

where in the next-to-last step I have dropped a term of order $|\xi^\alpha|^2$ The tensor transformation law for the metric tensor is then

$$\begin{aligned}g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^{\,\mu}}\frac{\partial x^\beta}{\partial x'^{\,\nu}} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu\xi^\alpha)(\delta^\beta_\nu - \partial_\nu\xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta})\\ &\approx \delta^\alpha_\mu\delta^\beta_\nu\eta_{\alpha\beta} - \partial_\mu\xi^\alpha\delta^\beta_\nu\eta_{\alpha\beta} - \delta^\alpha_\mu\partial_\nu\xi^\beta\eta_{\alpha\beta} + \delta^\alpha_\mu\delta^\beta_\nu h_{\alpha\beta} = \eta_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + h_{\mu\nu}\end{aligned} \tag{4.49}$$

where I have dropped terms of order $|\xi^\alpha|^2$ and also terms of order $|\xi^\alpha||h_{\mu\nu}|$ as being negligible. We see therefore that

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{4.50}$$

Now, suppose that we substitute this expression into the weak-field form of the Riemann tensor:

$$\begin{aligned}R'_{\alpha\beta\mu\nu} &= \tfrac{1}{2}(\partial'_\beta\partial'_\mu h'_{\alpha\nu} + \partial'_\alpha\partial'_\nu h'_{\beta\mu} - \partial'_\alpha\partial'_\mu h'_{\beta\nu} - \partial'_\beta\partial'_\nu h'_{\alpha\mu})\\ &= \tfrac{1}{2}(\partial_\beta\partial_\mu h'_{\alpha\nu} + \partial_\alpha\partial_\nu h'_{\beta\mu} - \partial_\alpha\partial_\mu h'_{\beta\nu} - \partial_\beta\partial_\nu h'_{\alpha\mu})\\ &= \tfrac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\beta\partial_\mu\partial_\alpha\xi_\nu - \partial_\beta\partial_\mu\partial_\alpha\xi_\nu + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\nu\partial_\beta\xi_\mu - \partial_\alpha\partial_\nu\partial_\mu\xi_\beta\\ &\quad - \partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\alpha\partial_\mu\partial_\beta\xi_\nu + \partial_\alpha\partial_\mu\partial_\nu\xi_\beta - \partial_\beta\partial_\nu h_{\alpha\mu} + \partial_\beta\partial_\nu\partial_\alpha\xi_\mu + \partial_\beta\partial_\nu\partial_\mu\xi_\alpha)\\ &= \tfrac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}) = R_{\alpha\beta\mu\nu}\end{aligned} \tag{4.51}$$

In going to the second line, I have dropped the primes on the partial derivatives because the difference between the two derivatives is of order $|\xi^\alpha|$, each derivative acts on a metric perturbation $h_{\mu\nu}$ and we are dropping terms of order $|\xi^\alpha||h_{\mu\nu}|$. In going to the third line, I have used equation 4.50. In going to the last line, I have noted that all of the terms involving ξ cancel in pairs: for each of the four subscript names α, β, μ, ν that ξ can have, we have two terms that cancel.

The point is that such a coordinate transformation does *not* change the value of any component of the Riemann tensor (to our order of approximation), which means that if our original coordinate system was a solution to the Einstein equation, then the new coordinate system will be as well. So, given any solution to the weak-field Einstein equation, we can generate an infinite *family* of solutions by using different (small but otherwise arbitrary transformations of the type $x'^{\alpha} = x^\alpha + \xi^\alpha$ where $|\xi^\alpha| \ll 1$.

At the deepest level, we see that this indeterminacy in the solution to the weak-field Einstein equation reflects our complete freedom to choose spacetime coordinates. But in the weak-field limit, we can *pretend* that $h_{\mu\nu}$ is not a perturbation in the metric but rather a tensor field that exists in a flat spacetime, in the same way that the electromagnetic four-potential A^μ is genuinely a tensor that exists in spacetime. We can then look at our freedom choose between equally valid solutions $h_{\mu\nu}$ as being analogous to the freedom we have in choosing valid electromagnetic potentials. You probably already know that if we transform the electromagnetic potentials by

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \text{and} \quad \phi' = \phi - \frac{\partial\lambda}{\partial t} \tag{4.52}$$

for any arbitrary scalar function $\lambda(t, x, y, z)$, we do not affect the physical electromagnetic fields $\vec{E}$ and $\vec{B}$, and that we call such transformations **gauge transformations** (for obscure historical reasons). Equation 4.50 represents completely analogous transformations of our metric perturbation $h_{\mu\nu}$, so because physicists are a lazy bunch, we still call them gauge transformations and any specific "nearly cartesian" coordinate choice for expressing $h_{\mu\nu}$ a **gauge**.

Just as the Lorenz gauge condition $\vec{\nabla} \cdot \vec{A} = -\partial\phi/\partial t$ (which is equivalent to $\partial_\mu A^\mu = 0$) simplifies the electromagnetic equations for the potentials, so our similar choice

$$0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h) \tag{4.53}$$

(which we also call the "Lorenz gauge condition" for the metric perturbation) greatly simplifies the Einstein equation. (By the way, this is "Lorenz," after the Danish physicist L. V. Lorenz, not "Lorentz" after the Dutch physicist H. A. Lorentz, whose name is attached to the Lorentz transformation. Particularly in the context of relativity, it is easy to confuse the two.) Now it is time to see that we can always make this choice.

Note that the transformation law for h is

$$h' = \eta^{\mu\nu}h'_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \eta^{\mu\nu}\partial_\mu\xi_\nu - \eta^{\mu\nu}\partial_\nu\xi_\mu = h - 2\eta^{\mu\nu}\partial_\nu\xi_\mu \tag{4.54}$$

So the transformation law for $H_{\mu\nu}$ must be

$$\begin{aligned} H'_{\mu\nu} &= h'_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h' = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu - \tfrac{1}{2}\eta_{\mu\nu}(h - 2\eta^{\alpha\beta}\partial_\alpha\xi_\beta) \\ &= H_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\xi_\beta \end{aligned} \tag{4.55}$$

Now, what we would like is to find what function ξ^α will take us from an arbitrary $H^{\mu\nu}$ to one satisfying $\partial'_\mu H'^{\mu\nu} = \partial_\mu H'^{\mu\nu} = 0$. Raising indices in the equation above, acting on both sides with ∂_μ, and setting the left side equal to zero implies that

$$0 = \partial_\mu H'^{\mu\nu} = \partial_\mu H^{\mu\nu} - \partial_\mu\partial^\mu\xi^\nu \cancel{- \partial_\mu\partial^\nu\xi^\mu} + \cancel{\partial^\nu\partial_\alpha\xi^\alpha} \tag{4.56}$$

We can do this by setting our transformation functions ξ^ν to be solutions of the equation $\Box^2\xi^\nu = \partial_\mu H^{\mu\nu}$. This is a simple set of four differential equations in ξ^ν. Mathematicians have thoroughly studied the differential operator $\Box^2 = -\partial^2/\partial t^2 - \nabla^2$, and have show that solutions to equations of the form $\Box^2 f = g$ always exist for well-defined driving functions g.

Indeed solutions f to the inhomogeneous differential equation $\Box^2 f = g$ actually represent *families* of solutions because if any given function f solves the equation, then so does $f + bf_0$, where f_0 is a solution to the homogeneous equation $\Box^2 f_0 = 0$. The point is that given a solution $H^{\mu\nu}$ that satisfies the Lorenz gauge condition, we can make further transformations ξ^ν satisfying $\Box^2\xi^\nu = 0$ without changing $\partial_\mu H^{\mu\nu}$ (and thus keeping the Lorenz condition intact). Choosing Lorenz gauge therefore does not exhaust our gauge freedom. We will find this very helpful when dealing with gravitational waves.

So, since we always can find a coordinate transformation that converts a solution $h^{\mu\nu}$ of the Einstein equation into one that satisfies the Lorenz condition $\partial_\mu H^{\mu\nu} = 0$ *after* we have found the solution, we can confidently require that we are looking for such solution *at the same time* we solve the Einstein equation.

Finally, I want to show you a revised version of the weak-field Einstein equation that is also useful. Suppose that we take equation 4.33 and contract over its free indices:

$$\begin{aligned} &\Box^2 \eta_{\mu\nu} h^{\mu\nu} = -16\pi G(\eta_{\mu\nu} T^{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu} T) = -16\pi G(T - 2T) \\ &\Rightarrow \quad \Box^2 h = +16\pi G T \end{aligned} \tag{4.57}$$

were I used $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^\mu_\mu = 4$. If we now subtract half of this from both sides if equation 4.33, we see that

$$\Box^2 H^{\mu\nu} = \Box^2(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} h) = -16\pi G(T^{\mu\nu} - \cancel{\tfrac{1}{2}\eta^{\mu\nu} T}) - \cancel{16\pi G \eta^{\mu\nu} T} = -16\pi G T^{\mu\nu} \tag{4.58}$$

Solving this simpler equation for $H^{\mu\nu}$ (subject to the Lorenz condition $\partial_\mu H^{\mu\nu} = 0$) therefore keeps us focused on *one* set of functions $H^{\mu\nu}$ and also allows us to use the straight stress-energy tensor on the right side. Once we have our solution for $H^{\mu\nu}$, then we can convert back to the real metric perturbation $h_{\mu\nu}$ by lowering indices and then using $h_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} H$. This will make our life simpler when we handle sources of gravitational waves.

4.6.1 Exercise: Is the Coordinate Transformation Small?

We found that we could enforce the Lorenz gauge condition to be true by finding transformation functions ξ^ν that solve $\Box^2 \xi^\nu = \partial_\mu H^{\mu\nu}$. But to be a valid gauge transformation, we also must have $\xi^\nu \ll 1$, so that the transformed versions of $h_{\mu\nu}$ don't violate the basic weak-field limit $|h_{\mu\nu}| \ll 1$. Why might we expect that solutions to $\Box^2 \xi^\nu = \partial_\mu H^{\mu\nu}$ would also satisfy this condition?

Homework Problems

4.1 Use the Diagonal Metric Worksheet to verify the values of $R_{rr}, R_{\theta\theta}$, and $R_{\phi\phi}$ stated above, and also check that off-diagonal Ricci components are all zero.

4.2 An object's four-acceleration is $\boldsymbol{a} \equiv d\boldsymbol{u}/d\tau$. As we saw in the last session $\boldsymbol{a} = 0$ for an object following a geodesic. Consider an object at rest at a Schwarzschild coordinate r. Such an object is *not* following a geodesic: one would have to exert an upward radial force (perhaps with a rocket engine) to hold it in place. The Schwarzschild components of the four-acceleration are

$$a^\mu = \left(\frac{d\boldsymbol{u}}{d\tau}\right)^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta \tag{4.59}$$

We have also seen that for an object at rest in Schwarzschild spacetime, we have $u^t = (1 - 2GM/r)^{-1/2}$ and $u^r = u^\theta = u^\phi = 0$, and that $\Gamma^r_{tt} = -2GM/r^2$ and other $\Gamma^\mu_{tt} = 0$. Calculate the acceleration's coordinate-independent magnitude $(\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Show that this goes to infinity as $r \to 2GM$. (This times the object's mass m will be the force required to hold the object in place.)

4.3 Consider the weak field solution for a spherical star that we developed during the session.

- **(a)** Show that the spatial part of the metric in this solution is $ds^2 = (1 + 2GM/r)(dx^2 + dy^2 + dz^2)$.
- **(b)** Consider the coordinate transformations $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ (the same transformations that we would use to go from spherical to cartesian coordinates in flat-spacetime). Take the differentials of these expressions and substitute them into the above to show that in r, θ, ϕ coordinates, the spatial part of the metric is $ds^2 = (1 + 2GM/r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$.
- **(c)** Note that r here is *not* a circumferential coordinate since the circumference of the equator is $2\pi(1 + 2GM/r)^{1/2}r$. So define a truly circumferential coordinate $r_c \equiv (1 + 2GM/r)^{1/2}r$. Show that $dr = dr_c$ if we ignore terms of order $2GM/r$ ($= h_{tt}$ in this case).
- **(d)** Show then that the spatial part of the metric becomes $ds^2 = (1+2GM/r)dr_c^2 + r_c^2 d\theta^2 + r_c^2 \sin^2\theta d\phi^2$.
- **(e)** Show that this is equivalent to $ds^2 = (1 + 2GM/r_c)dr_c^2 + r_c^2 d\theta^2 + r_c^2 \sin^2\theta d\phi^2$ if we drop terms that are second order in $2GM/r = h_{tt}$.
- **(f)** Use the binomial approximation to show that the last metric is equivalent to the spatial Schwarzschild metric to the same order.

4.4 Show that the basic vector identities $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ and $\vec{\nabla} \times \vec{\nabla} f = 0$ (where $\vec{F}$ and f are arbitrary vector and scalar fields, respectively) and the definitions of $\vec{E}_G$ and $\vec{B}_G$ imply Gauss's law for $\vec{B}_G$ and Faraday's law for $\vec{E}_G$ and $\vec{B}_G$.

4.5 We can get a first step of insight into how one might derive $\vec{a} = \vec{E}_G + \vec{v} \times 4\vec{B}_G$ (equation 4.46) as follows. Start with the geodesic equation

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}u^\mu u^\nu \tag{4.60}$$

Assume that the particle is moving so slowly that $\tau \approx t$, which basically means that we are ignoring terms of order v^2 in the particle's speed v. This will also mean that $u^t \approx 1$ and $u^i \approx v^i$. (Technically, u^t also differs from 1 by terms of order $h_{\mu\nu}$, but since the Christoffel symbol is already of that order, neglecting those terms in u^t will only yield negligible errors of order $|h_{\mu\nu}|^2$). Let's assume that the particle is moving purely in the x-direction. The geodesic equation then becomes

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{tt} - \Gamma^i_{tx}v^x \tag{4.61}$$

Ignore first term on the right: that will become $\vec{E}_G$, but the argument is a bit subtle. Focus instead on the velocity-dependent part. Assume that the fields are static, so that all time derivatives of $h_{\mu\nu}$ are zero. Use the definition of the Christoffel symbol and the definitions of $\vec{A}_G$ and $\vec{B}_G$ to show that this part of the acceleration has components equal to the components of $\vec{v} \times 4\vec{B}_G$.

4.6 Suppose that in a certain region of spacetime, we have $H^{\mu\nu} = 0$ except for $H^{tt} = Br^2 + C$, where B and C are constants (B has units of m^{-2}) and $r^2 \equiv x^2 + y^2 + z^2$.

- **(a)** Show that $H^{\mu\nu}$ satisfies the Lorenz condition.
- **(b)** Use equation 4.58 to argue that this could be a solution to the weak-field Einstein equation *inside* a spherical star centered on the origin whose fluid has a *uniform and constant* energy density ρ, negligible pressure, and has zero bulk velocity. Also determine how the constant B must be related to ρ.
- **(c)** Define $m(r) = \frac{4}{3}\pi r^3 \rho =$ the Newtonian mass-energy enclosed by the radius r. Calculate all of the metric components of $g_{\mu\nu}$ as a function of r in terms of $m(r)$ and C.
- **(d)** At the star's surface $r = R$, the solution must match the exterior solution that we developed during the session (note that $m(R) = M$. Use this to determine the value of C and right out the final metric equation for the star's interior in these coordinates.
- **(e)** Sketch or plot a graph of h_{tt} (in terms of GM/R) as a function of r from $r = 0$ to $r = 4R$.
- **(f)** What condition must be satisfied if all points inside such a star are to satisfy the weak-field condition that $|h_{\mu\nu}| \ll 1$? Is this plausible for normal matter?

Notes

[1]Folsing, *Albert Einstein*, (trans. Osers), Penguin 1997, pp. 384-385.

[2]Everitt, et al. (2011). "Gravity Probe B: Final Results of a Space Experiment to Test General Relativity". *Physical Review Letters,* **106** (22): 221101.