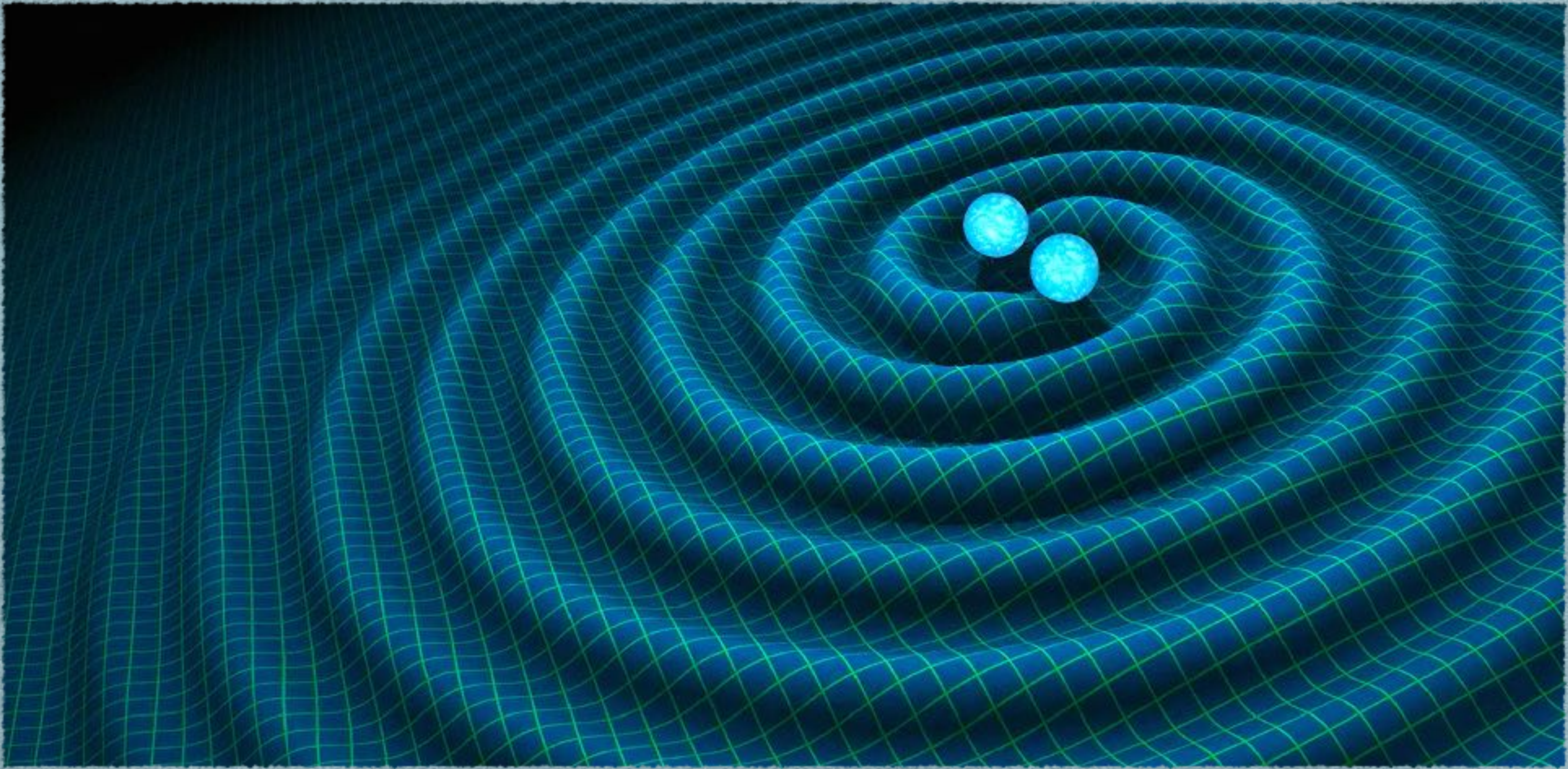

General Relativity and Gravitational Waves:

Session 4. Solving the Einstein Equation

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Overview of this session:

4.2 The Schwarzschild Solution

4.3 Interpreting the Schwarzschild Solution

4.4 The Weak-Field Approximation

4.5 Gravitomagnetism

4.6 Gauge Freedom

The Schwarzschild Solution

We solve the Einstein equation like we solve any differential equation: (1) guess a solution, (2) solve for one or two undetermined parameters.

Assume two basic symmetries:

1. Spherical symmetry
2. Time-independence (star is static)

We want to solve the empty-space Einstein equation

$$R_{\mu\nu} = 0$$

for the vacuum region outside the star.

The Schwarzschild Solution

Imagine surrounding star with nested spheres. Spherical symmetry implies that we ought to be able to use ordinary spherical coordinates on each surface. The metric (for a spherical coordinate basis): $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

This implicitly defines the meaning of r: the arc length of an equatorial circle ($\theta = \pi/2$, $d\theta = 0$) is

$$s = \int ds = \int r\, d\phi = 2\pi r \quad \Rightarrow \quad r = \frac{s}{2\pi}$$

We can make the angular coordinates line up on each sphere by making the lines of constant θ, ϕ be radial between spheres: this means $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ are mutually perpendicular.

The Schwarzschild Solution

Our metric so far is therefore $ds^2 = g_{rr}\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

What about a term like $g_{t\phi}\, dt\, d\phi$? This would imply that $+d\phi$ physically different than $-\,d\phi$: this is not consistent with spherical symmetry (or time reversal symmetry). A term like $g_{rt}\, dr\, dt$ would also violate time reversal symmetry. So we *ought* to be able to find a solution of the form

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

Moreover, spherical symmetry and time-independence suggest that the unknown components can depend at most on r.

The Diagonal Metric Worksheet

Consider the following general diagonal metric:

$$ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2$$

where dx^0, dx^1, dx^2, and dx^3 are completely arbitrary coordinates and A, B, C, and D are arbitrary functions of any or all of the coordinates. This worksheet (adapted from results listed in Rindler, *Essential Relativity*, 2/e, Springer-Verlag, 1977) allows you to quickly calculate the components of $\Gamma^{\alpha}_{\mu\nu}$ and $R_{\mu\nu} \equiv +R^{\alpha}{}_{\mu\alpha\nu}$ for any specific special case of such a metric. In this worksheet, I use the following shorthand notation:

$$A_0 \equiv \frac{\partial A}{\partial x^0}, \quad B_{12} \equiv \frac{\partial^2 B}{\partial x^1 \partial x^2}, \text{ and so on.}$$

To use this worksheet, start by crossing out each tabulated term that is zero for the specific metric in question. For the remaining terms, write the term's value in the space above that term. For the Ricci tensor components, you can then gather the terms in the space provided at the bottom. To adapt this worksheet to smaller dimensional spaces or spacetimes, treat the metric components corresponding to any nonexistent coordinates as if they had the value 1 and the remaining metric components as being independent of the nonexistent coordinates.

The Diagonal Metric Worksheet

$$\frac{1}{2B}\frac{d^2A}{dr^2}$$

$$R_{00} = 0 \quad + \frac{1}{2B}A_{11} \quad + \cancel{\frac{1}{2C}A_{22}} \quad + \cancel{\frac{1}{2D}A_{33}}$$

$$+ \quad 0 \quad - \frac{1}{2B}\cancel{B_{00}} \quad - \frac{1}{2C}\cancel{C_{00}} \quad - \frac{1}{2D}\cancel{D_{00}}$$

$$+ \quad 0 \quad + \frac{1}{4B^2}\cancel{B_0^2} \quad + \frac{1}{4C^2}\cancel{C_0^2} \quad + \frac{1}{4D^2}\cancel{D_0^2}$$

$$+ \quad 0 \quad + \frac{1}{4AB}\cancel{A_0}B_0 \quad + \frac{1}{4AC}\cancel{A_0}C_0 \quad + \frac{1}{4AD}\cancel{A_0}D_0$$

$$-\frac{1}{4BA}\left(\frac{dA}{dr}\right)^2 \quad -\frac{1}{4B^2}\frac{dA}{dr}\frac{dB}{dr} \quad + \frac{1}{4Br^2}\frac{dA}{dr}(2r) \quad + \frac{1}{4Br^2\cancel{\sin^2\theta}}\frac{dA}{dr}\,2r\cancel{\sin^2\theta}$$

$$- \frac{1}{4BA}A_1A_1 \quad - \frac{1}{4B^2}A_1B_1 \quad + \frac{1}{4BC}A_1C_1 \quad + \frac{1}{4BD}A_1D_1$$

$$- \frac{1}{4CA}\cancel{A_2}A_2 \quad + \frac{1}{4CB}\cancel{A_2}B_2 \quad - \frac{1}{4C^2}\cancel{A_2}C_2 \quad + \frac{1}{4CD}\cancel{A_2}D_2$$

$$- \frac{1}{4DA}\cancel{A_3}A_3 \quad + \frac{1}{4DB}\cancel{A_3}B_3 \quad + \frac{1}{4DC}\cancel{A_3}C_3 \quad - \frac{1}{4D^2}\cancel{A_3}D_3$$

$$\boxed{R_{00} = \frac{1}{2B}\left[\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right]}$$

The Schwarzschild Solution

The worksheet yields

$$R_{tt} = \frac{1}{2B}\left[\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right]$$

$$R_{rr} = \frac{1}{2A}\left[-\frac{d^2A}{dr^2} + \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2A}{Br}\frac{dB}{dr}\right]$$

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B}, \qquad R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$

Combining the first two:

$$0 = 2BR_{tt} + 2AR_{rr} = \frac{2}{r}\frac{dA}{dr} + \frac{2A}{Br}\frac{dB}{dr}$$

$$0 = B\frac{dA}{dr} + A\frac{dB}{dr} = \frac{d}{dr}(AB) \quad \Rightarrow \quad AB = \text{constant}$$

The Schwarzschild Solution

Now substitute $B = 1/A$ and $0 = B\dfrac{dA}{dr} + A\dfrac{dB}{dr}$ into

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B}$$ to get (after a bit of work):

$$0 = -r\frac{dA}{dr} + 1 - A$$

$$\Rightarrow \quad 1 = r\frac{dA}{dr} + A = \frac{d}{dr}(rA) \quad \Rightarrow \quad r = rA + K \quad \Rightarrow \quad A = 1 - \frac{K}{r}$$

We saw that $K = 2GM$ gave the right acceleration in the Newtonian limit, so $A = 1 - 2GM/r$ and $B = 1/A = (1 - 2GM/r)^{-1}$.

Exercise: Fill in the missing steps above.

Interpreting the Schwarzschild Solution

r is not a radial coordinate! The distance between r_0 and r_1 is

$$\Delta s = \int ds = \int_{r_0}^{r_1} \frac{dr}{1 - 2GM/r} = \left[r\sqrt{1 - \frac{2GM}{r}} + 2GM \tanh^{-1}\sqrt{1 - \frac{2GM}{r}} \right]_{r_0}^{r_1}$$

t is not what a clock at rest measures:

$$\Delta\tau = \int \sqrt{-ds^2} = \int \sqrt{1 - \frac{2GM}{r}}\, dt = \left(1 - \frac{2GM}{r}\right) \Delta t$$

Light flashes follow worldlines such that $ds = 0$, so

$$0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/R} \quad \Rightarrow \quad \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{r}\right)^2$$

$$\Rightarrow \quad \frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right)$$

Interpreting the Schwarzschild Solution

We can set up a lattice around the star with spherical surfaces lashed together by radial girders, and a t-meter at every intersection, which is turned up to match clocks at infinity, and which is synchronized with the clock at the same θ, ϕ by exchanging light signals.

But this scheme falls apart as r approaches $2GM$:

1. We cannot determine radial distances for $r < 2GM$
2. A clock at rest at $r = 2GM$ measures no time (so can't synchronize)
3. Such a clock would need to be constructed of photons

The basic issue is that for $r < 2GM$, $g_{rr} < 0$ and $g_{tt} > 0$

1. This means that the r coordinate becomes a time coordinate!
2. Any object's future is $r = 0$!
3. So our whole lattice scheme is impossible inside $r = 2GM$

Exercise: Kruskal-Szekeres

The Kruskal-Szekeres coordinate system is an alternative solution to the empty-space Einstein equation in the spherically symmetric case. The Kruskal-Szekeres metric is

$$ds^2 = -\frac{32(GM)^3}{r}e^{-r/2GM}(dv^2 - du^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.18}$$

where r is not a coordinate here but is a shorthand for a function $r(v,u)$ implicitly defined by

$$\left(\frac{r}{2GM} - 1\right)e^{r/2GM} = u^2 - v^2 \tag{4.19}$$

Note that the components of this metric do *not* behave badly at $r = 2GM$ (which corresponds to events where $u^2 = v^2$).

(a) What is the time coordinate in this metric? Is it a time coordinate for all values of r?

(b) What kind of worldline does a particle at fixed r follow in u, v coordinates? Is r fixed if u is fixed? If v is fixed?

(c) Argue that the value of the function r still corresponds to the circumference of an equatorial ($\theta = \pi/2$) circle divided by 2π but now evaluated for fixed u and v (instead of fixed r and t as in Schwarzschild coordinates). Is r fixed if u *and* v are fixed?

Weak-Field Approximation: Basic Definitions

The weak-field approximation assumes "nearly cartesian" coordinates t, x, y, z, for which:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \quad h_{\mu\nu} = h_{\nu\mu} \text{ and } |h_{\mu\nu}| \ll 1$$

We will drop terms of order $|h_{\mu\nu}|^2$ and higher.

The inverse metric is: $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$
because:

$$\begin{aligned}
g^{\mu\nu}g_{\nu\sigma} &= (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\sigma} + h^{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} - h^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + h^{\mu\nu}h_{\nu\sigma} \\
&= \delta^\mu_\sigma - \eta_{\nu\sigma}(\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}) + \eta^{\mu\nu}h_{\nu\sigma} + (\text{dropped}) \\
&= \delta^\mu_\sigma - \eta^{\mu\alpha}(\eta_{\sigma\nu}\eta^{\nu\beta})h_{\alpha\beta}) + \eta^{\mu\nu}h_{\nu\sigma} \\
&= \delta^\mu_\sigma - \eta^{\mu\alpha}\delta^\beta_\sigma h_{\alpha\beta} + \eta^{\mu\nu}h_{\nu\sigma} = \delta^\mu_\sigma - \cancel{\eta^{\mu\alpha}h_{\alpha\sigma}} + \cancel{\eta^{\mu\nu}h_{\nu\sigma}} = \delta^\mu_\sigma
\end{aligned}$$

Weak-Field Approximation: Raising/Lowering, Riemann tensor

This means we can raise or lower any quantity of order $h_{\mu\nu}$ with $\eta_{\mu\nu}$:

$$h^{\mu}{}_{\nu} \equiv g^{\mu\alpha} h_{\alpha\nu} \approx (\eta^{\mu\alpha} - h^{\mu\alpha}) h_{\alpha\nu} = \eta^{\mu\alpha} h_{\alpha\nu} + (\text{dropped})$$

Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \approx \tfrac{1}{2} \eta^{\alpha\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})$$

Riemann tensor ("inside togetherness is positive")

$$R_{\alpha\beta\mu\nu} = \tfrac{1}{2} (\partial_\beta \partial_\mu h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\nu h_{\alpha\mu})$$

Einstein equation:

$$R_{\beta\nu} = 8\pi G (T_{\beta\nu} - \tfrac{1}{2} g_{\beta\nu} T) \quad \text{where } T \equiv g_{\mu\nu} T^{\mu\nu}$$

where Ricci tensor is

$$R_{\beta\nu} \equiv g^{\alpha\mu} R_{\alpha\beta\mu\nu} \approx \tfrac{1}{2} \eta^{\alpha\mu} (\partial_\beta \partial_\mu h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\nu h_{\alpha\mu})$$

Weak-Field Approximation: Trace-Reversed Metric Perturbation

Define: $H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$

"Trace-reversed" because

$$H \equiv \eta^{\mu\nu}H_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \eta^{\mu\nu}\eta_{\mu\nu}h = h - \tfrac{1}{2}\delta^{\mu}_{\mu}h = h - 2h = -h$$

To go back: $h_{\mu\nu} = H_{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}h = H_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}H$

The Ricci tensor becomes (with $\Box^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial^2/\partial_t^2 + \nabla^2$):

$$\begin{aligned}
R_{\beta\nu} &\approx \tfrac{1}{2}(\partial_\beta\partial_\mu\eta^{\alpha\mu}[H_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu}H] + \partial_\alpha\partial_\nu\eta^{\alpha\mu}[H_{\beta\mu} - \tfrac{1}{2}\eta_{\beta\mu}H] - \eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\beta\partial_\nu H) \\
&= \tfrac{1}{2}(\partial_\beta\partial_\mu[H^{\mu}{}_{\nu} - \cancel{\tfrac{1}{2}\delta^{\mu}_{\nu}H}] + \partial_\alpha\partial_\nu[H^{\alpha}{}_{\beta} - \cancel{\tfrac{1}{2}\delta^{\alpha}_{\beta}H}] - \partial^\mu\partial_\mu h_{\beta\nu} + \cancel{\partial_\beta\partial_\nu H}) \\
&= \tfrac{1}{2}(\partial_\beta\partial_\mu[\eta_{\nu\sigma}H^{\mu\sigma}] + \partial_\alpha\partial_\nu[\eta_{\beta\sigma}H^{\alpha\sigma}] - \Box^2 h_{\beta\nu})
\end{aligned}$$

Einstein equation:

$$\Box^2 h_{\beta\nu} - \eta_{\nu\sigma}\partial_\beta\partial_\mu H^{\mu\sigma} - \eta_{\beta\sigma}\partial_\alpha\partial_\nu H^{\alpha\sigma} = -16\pi G(T_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}T)$$

Weak-Field Approximation: Solving the Einstein equation

Einstein equation:

$$\Box^2 h_{\beta\nu} - \eta_{\nu\sigma}\partial_\beta\partial_\mu H^{\mu\sigma} - \eta_{\beta\sigma}\partial_\alpha\partial_\nu H^{\alpha\sigma} = -16\pi G(T_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}T)$$

becomes equivalent to simultaneously solving:

$$\Box^2 h^{\beta\nu} = -16\pi G(T^{\beta\nu} - \tfrac{1}{2}\eta^{\beta\nu}T) \quad \text{and} \quad 0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h)$$

We solve this in analogy to the way that we solve

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = -\frac{\rho_c}{\varepsilon_0} = -4\pi k\rho_c$$

The solution is:

$$h^{\mu\nu}(t, \vec{R}) = 4G\int_{\text{src}} \frac{\overline{T}^{\mu\nu}(t-s, \vec{r})\, dV}{s} \quad \text{where} \quad \overline{T}^{\mu\nu} \equiv T^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}T$$

Weak-Field Approximation: Exercise: A static spherical star

Consider a static spherical star with total mass M. Assume that pressure is negligible, so that its only nonzero stress-energy component is $T^{tt} = \rho$, where ρ is a function of radius alone.

(a) Show that $\overline{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T$ has components $\overline{T}^{tt} = \overline{T}^{xx} = \overline{T}^{yy} = \overline{T}^{zz} = \frac{1}{2}\rho$ at every point.

(b) We know from electrostatics that the potential outside a spherical source is the same as that for particle with same charge located at the source's center. Using that analogy, find all components of $h^{\mu\nu}$ at a point outside the star that is a distance r from the star's center.

(c) What is $H^{\mu\nu}$ for this solution? Show that $\partial_\mu H^{\mu\nu} = 0$.

(d) What is g_{tt} outside the star? Does this look familiar? (Don't forget to lower the indices on $h^{\mu\nu}$.)

Gravitomagnetism: Potentials

Define **gravitoelectric** and **gravitovector** potentials:

$$\Phi_G \equiv -\tfrac{1}{8}(h^{tt} + h^{xx} + h^{yy} + h^{zz}) \quad \text{and} \quad A^i_G \equiv -\tfrac{1}{4}h^{ti}$$

The time component of our coordinate condition implies that

$$0 = \partial_\mu(h^{\mu t} - \tfrac{1}{2}\eta^{\mu t}h) = \partial_t h^{tt} + \partial_i h^{it} - \tfrac{1}{2}(-1)\partial_t(-h^{tt} + h^{xx} + h^{yy} + h^{zz})$$

$$= \partial_i h^{it} + \tfrac{1}{2}\partial_t(h^{tt} + h^{xx} + h^{yy} + h^{zz}) = -4\vec{\nabla} \cdot \vec{A}_G - 4\partial_t \Phi_G$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{A}_G = -\frac{\partial \Phi_G}{\partial t} \qquad \text{(like Lorenz gauge condition for EM)}$$

Gravitomagnetism: "Electric Field"

Define **gravitoelectric field:** $\vec{E}_G \equiv -\vec{\nabla}\Phi_G - \partial\vec{A}_G/\partial t$. Then

$$\vec{\nabla}\cdot\vec{E}_G = \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi_G - \frac{\partial\vec{A}_G}{\partial t}\right) = -\nabla^2\Phi_G - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}_G\right) = -\Box^2\Phi_G$$

$$= +\tfrac{1}{8}\Box^2(h^{tt} + h^{xx} + h^{yy} + h^{zz})$$

But by the weak-field Einstein equation:

$$\begin{aligned}\vec{\nabla}\cdot\vec{E}_G &= -2\pi G(T^{tt} + T^{xx} + T^{yy} + T^{zz} + \tfrac{1}{2}T - \tfrac{1}{2}T - \tfrac{1}{2}T - \tfrac{1}{2}T) \\ &= -2\pi G(T^{tt} + T^{xx} + T^{yy} + T^{zz} - [-T^{tt} + T^{xx} + T^{yy} + T^{zz}]) \\ &= -4\pi G T^{tt} = -4\pi G\rho \qquad \text{(!)}\end{aligned}$$

Gravitomagnetism: "Magnetic Field"

Define **gravitomagnetic field:** $\vec{B}_G = \vec{\nabla} \times \vec{A}_G$. Then

$$\vec{\nabla} \times \vec{B}_G = \vec{\nabla} \times \vec{\nabla} \times \vec{A}_G = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}_G) - \nabla^2 A_G = \vec{\nabla}\left(-\frac{\partial \Phi_G}{\partial t}\right) - \nabla^2 A_G$$

If we add and subtract $\partial^2 \vec{A}_G / \partial t^2$ and use the Einstein equation, we get

$$\vec{\nabla} \times \vec{B}_G = \frac{\partial}{\partial t}\left(\vec{\nabla}\Phi_G - \frac{\partial \vec{A}_G}{\partial t}\right) + \frac{\partial^2 \vec{A}_G}{\partial t^2} - \nabla^2 A_G = \frac{\partial \vec{E}_G}{\partial t} - \Box^2 \vec{A}_G$$

$$= \frac{\partial \vec{E}_G}{\partial t} + \tfrac{1}{4}\Box^2 h^{ti} = \frac{\partial \vec{E}_G}{\partial t} - 4\pi G(T^{ti} - \tfrac{1}{2}\eta_{ti} T) = \frac{\partial \vec{E}_G}{\partial t} - 4\pi G(\vec{J} + 0)$$

where $\vec{J}$ is the energy flux in the three spatial directions.

Gravitomagnetism: Maxwell's Equations!

The definitions of the fields in terms of the potentials imply the other Maxwell equations in the usual way (vector identities). So:

$$\vec{\nabla} \cdot \vec{E}_G = -4\pi G\rho$$

$$\vec{\nabla} \times \vec{B}_G - \frac{\partial \vec{E}_G}{\partial t} = -4\pi G\vec{J}$$

$$\vec{\nabla} \cdot \vec{B}_G = 0$$

$$\vec{\nabla} \times \vec{E}_G + \frac{\partial \vec{B}_G}{\partial t} = 0$$

(Note minus signs: gravity is *attractive* for like masses, and we use a left-hand rule for magnetic fields from currents.) Applies quite generally for weak fields! Under more limited conditions:

$$\vec{a} \approx \vec{E}_G + \vec{v} \times 4\vec{B}_G$$

Exercise: Basic gravitomagnetism

(a) Use the gravitomagnetic analogy to argue that a particle falling initially radially in the equatorial plane of a spinning object does indeed experience an acceleration component that deflects in the direction of the object's spin. (Don't forget the left-hand rule!)

(b) Imagine a particle in a circular orbit in the equatorial plane of a spinning object. Suppose the particle is orbiting in the same direction as the object is rotating. Will the period of the particle's orbit be affected by the object's spin? If so, will it be longer or shorter than if the object were not spinning?

(c) Imagine an uncharged particle of dust is moving initially parallel to a relativistic stream of particles (perhaps a jet from a quasar). Will gravitomagnetic effects repel or attract it to the stream? Are those effects likely to be as large as the basic gravitoelectric attraction it experiences toward the stream?

Gauge Freedom

We *required* that $0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)$. Why can we?

Our "nearly cartesian" coordinates are not completely defined: *lots* of coordinate systems can satisfy $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $|h_{\mu\nu}| \ll 1$.

Specifically, we can have coordinate transformations such that

$$x'^{\,\alpha} = x^\alpha + \xi^\alpha \quad \text{where} \quad \xi^\alpha = \xi^\alpha(t,x,y,z) \text{ and } \quad |\xi^\alpha| \ll 1$$

Transformation partials are:

$$\frac{\partial x^\beta}{\partial x'^{\,\alpha}} = \frac{\partial}{\partial x'^{\,\alpha}}\left(x'^{\,\beta} - \xi^\beta\right) = \delta^\beta_\alpha - \frac{\partial \xi^\beta}{\partial x'^{\,\alpha}} = \delta^\beta_\alpha - \frac{\partial x^\sigma}{\partial x'^{\,\alpha}}\frac{\partial \xi^\beta}{\partial x^\sigma}$$

$$= \delta^\beta_\alpha - \left(\delta^\sigma_\alpha - \frac{\partial \xi^\sigma}{\partial x'^{\,\alpha}}\right)\frac{\partial \xi^\beta}{\partial x^\sigma} \approx \delta^\beta_\alpha - \delta^\sigma_\alpha \frac{\partial \xi^\beta}{\partial x^\sigma} = \delta^\beta_\alpha - \partial_\alpha \xi^\beta$$

Gauge Freedom

So our transformed metric is:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^{\,\mu}} \frac{\partial x^\beta}{\partial x'^{\,\nu}} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta})$$

$$\approx \delta^\alpha_\mu \delta^\beta_\nu \eta_{\alpha\beta} - \partial_\mu \xi^\alpha \delta^\beta_\nu \eta_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \xi^\beta \eta_{\alpha\beta} + \delta^\alpha_\mu \delta^\beta_\nu h_{\alpha\beta} = \eta_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h_{\mu\nu}$$

We can consider this a transformation of the perturbation:

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Substitute this into the Riemann tensor:

$$\begin{aligned}
R'_{\alpha\beta\mu\nu} &= \tfrac{1}{2}(\partial'_\beta \partial'_\mu h'_{\alpha\nu} + \partial'_\alpha \partial'_\nu h'_{\beta\mu} - \partial'_\alpha \partial'_\mu h'_{\beta\nu} - \partial'_\beta \partial'_\nu h'_{\alpha\mu}) \\
&= \tfrac{1}{2}(\partial_\beta \partial_\mu h'_{\alpha\nu} + \partial_\alpha \partial_\nu h'_{\beta\mu} - \partial_\alpha \partial_\mu h'_{\beta\nu} - \partial_\beta \partial_\nu h'_{\alpha\mu}) \\
&= \tfrac{1}{2}(\partial_\beta \partial_\mu h_{\alpha\nu} - \partial_\beta \partial_\mu \partial_\alpha \xi_\nu - \partial_\beta \partial_\mu \partial_\alpha \xi_\nu + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\nu \partial_\beta \xi_\mu - \partial_\alpha \partial_\nu \partial_\mu \xi_\beta \\
&\quad - \partial_\alpha \partial_\mu h_{\beta\nu} + \partial_\alpha \partial_\mu \partial_\beta \xi_\nu + \partial_\alpha \partial_\mu \partial_\nu \xi_\beta - \partial_\beta \partial_\nu h_{\alpha\mu} + \partial_\beta \partial_\nu \partial_\alpha \xi_\mu + \partial_\beta \partial_\nu \partial_\mu \xi_\alpha) \\
&= \tfrac{1}{2}(\partial_\beta \partial_\mu h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\nu h_{\alpha\mu}) = R_{\alpha\beta\mu\nu}
\end{aligned}$$

Gauge Freedom

Analogous to gauge transformations in EM theory:

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \text{and} \quad \phi' = \phi - \frac{\partial\lambda}{\partial t}$$

Lorenz gauge in EM: $\vec{\nabla} \cdot \vec{A} = -\partial\phi/\partial t$ (equivalent to $\partial_\mu A^\mu = 0$)

Lorenz gauge in GR: $0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)$

But does our transformation give us the freedom to choose this gauge?

Gauge Freedom

The transformation law for the basic metric perturbation is:

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

This implies that

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\partial_\mu \xi_\nu - \eta^{\mu\nu}\partial_\nu \xi_\mu = h - 2\eta^{\mu\nu}\partial_\nu \xi_\mu$$

So the transformation law for the trace-reversed perturbation is

$$\begin{aligned} H'_{\mu\nu} &= h'_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \tfrac{1}{2}\eta_{\mu\nu}(h - 2\eta^{\alpha\beta}\partial_\alpha \xi_\beta) \\ &= H_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha \xi_\beta \end{aligned}$$

We want to find the transformation to take us to $\partial'_\mu H'^{\,\mu\nu} = \partial_\mu H'^{\,\mu\nu} = 0$.
The transformation above implies that

$$0 = \partial_\mu H'^{\,\mu\nu} = \partial_\mu H^{\mu\nu} - \partial_\mu \partial^\mu \xi^\nu - \cancel{\partial_\mu \partial^\nu \xi^\mu} + \cancel{\partial^\nu \partial_\alpha \xi^\alpha}$$

That is, we want to solve $\Box^2 \xi^\nu = \partial_\mu H^{\mu\nu}$.

Gauge Freedom

Mathematicians have shown that solutions f to $\Box^2 f = g$ exist for well-defined g. Indeed, we have families of solutions because if f is a solution, then so is $f + bf_0$, where f_0 solves $\Box^2 f_0 = 0$.

So given a solution that satisfies Lorenz gauge, we can make *further* transformations satisfying $\Box^2 \xi^\nu = 0$ and still be in Lorenz gauge. We'll use this later.

Einstein equation for H

Basic Einstein equation:

$$\Box^2 h^{\beta\nu} = -16\pi G(T^{\beta\nu} - \tfrac{1}{2}\eta^{\beta\nu}T) \quad \text{and} \quad 0 = \partial_\mu H^{\mu\nu} = \partial_\mu(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h)$$

Contract the Einstein equation over free indices:

$$\Box^2 \eta_{\mu\nu}h^{\mu\nu} = -16\pi G(\eta_{\mu\nu}T^{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}T) = -16\pi G(T - 2T)$$

$$\Rightarrow \quad \Box^2 h = +16\pi G T$$

Subtract this from both sides of the basic Einstein equation to get

$$\Box^2 H^{\mu\nu} = \Box^2(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}h) = -16\pi G(T^{\mu\nu} - \cancel{\tfrac{1}{2}\eta^{\mu\nu}T}) - \cancel{16\pi G\eta^{\mu\nu}T}$$

$$= -16\pi G T^{\mu\nu}$$

We can solve this simpler equation and then revert back to

$$h_{\mu\nu} = H_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}H$$

Exercise:
Is the coordinate transformation small?

We found that we could enforce the Lorenz gauge condition to be true by finding transformation functions ξ^ν that solve $\Box^2 \xi^\nu = \partial_\mu H^{\mu\nu}$. But to be a valid gauge transformation, we also must have $\xi^\nu \ll 1$, so that the transformed versions of $h_{\mu\nu}$ don't violate the basic weak-field limit $|h_{\mu\nu}| \ll 1$. Why might we expect that solutions to $\Box^2 \xi^\nu = \partial_\mu H^{\mu\nu}$ would also satisfy this condition?