General Relativity and Gravitational Waves

Session 5: Gravitational Waves

5.1 Overview of this Session

In the last session, we began exploring the weak-field approximation to general relativity and our gauge freedom in choosing the solutions to the weak-field Einstein equation. In this final session we will explore solutions to the weak-field Einstein equation that describe gravitational waves and how they are generated.

An overview of this session's sections follows:

5.2 Transverse-Traceless Gauge. Not all wavelike solutions of the weak-field Einstein equation are actual waves. This section discusses how we can discover which waves are real and which are "fake," and how going to the so-called "transverse-traceless" gauge focuses our attention on the real physical of gravitational waves and how they affect matter.

5.3 Generating Gravitational Waves. In this section, we will see that for *small, weak,* and *slow* sources, we can link the gravitational waves generated to the double time derivative of the source's reduced quadrupole moment tensor.

5.4 Gravitational Wave Energy. This section explores the tricky issue of how we can determine the "energy" that a gravitational wave carries.

5.5 Source Luminosities. In order to calculate the gravitational-wave luminosity of sources, we need to be able to determine the transverse-traceless components of the metric perturbation for waves moving in arbitrary directions. This section devises such a method and develops a formula for a source's luminosity.

5.6 Gravitational Waves from Binary Stars. This section talks about how we can specifically calculate the gravitational radiation from binary star systems (including black-hole binaries), which are the main (known) astrophysical source of detectable gravitational waves. This will include discussion of the proposed LISA detector and post-newtonian approximations for gravitational waves from such sources.

5.2 The Transverse-Traceless Gauge.

We begin where we left off last time. We seek to solve the weak-field Einstein equation in nearly cartesian coordinates, where it has the form

$$\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu} \quad \text{where } H^{\mu\nu} \equiv h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} h \quad \text{and } g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with } |h_{\mu\nu}| \ll 1 \tag{5.1}$$

subject to the Lorenz gauge condition

$$\partial_\mu H^{\mu\nu} = 0 \tag{5.2}$$

that restricts our choice of coordinates. But recall also that we have some remaining freedom to choose coordinates: we can apply coordinate transformations of the form

$$x'^{\alpha} = x^\alpha + \xi^\alpha \quad \text{where } |\xi^\alpha| \ll 1 \quad \text{and} \quad \Box^2 \xi^\alpha = 0 \tag{5.3}$$

Under such a coordinate transformation, we found that

$$H'^{\mu\nu} = H^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\mu \xi^\nu + \eta^{\mu\nu}\partial_\alpha \xi^\alpha \tag{5.4}$$

Now, in empty space, the weak-field Einstein equation has the form $\Box^2 H^{\mu\nu} = 0$. Since $\Box^2 f = 0$ for any function f is the wave equation, we can immediately see that we are going to have gravitational wave solutions. Let's attempt a plane-wave solution of the form

$$H^{\mu\nu} = A^{\mu\nu} \cos k_\sigma x^\sigma = A^{\mu\nu} \cos(\vec{k} \cdot \vec{r} - \omega t) \tag{5.5}$$

where $A^{\mu\nu}$ is a constant matrix and k_σ is a constant covector with components $k_t = -\omega, k_x, k_y, k_z$. Such a wave is a plane wave whose crests are perpendicular to the $\vec{k}$ direction and which move in the $+\vec{k}$ direction with phase speed $v = \omega/k$. The Einstein equation, the Lorenz gauge condition, and the symmetry of $H^{\mu\nu}$ (which follows from the symmetry of $h_{\mu\nu}$) imply that

$$\begin{aligned} &\text{Einstein equation:} && \Rightarrow \quad k^\alpha k_\alpha = 0 && (5.6)\\ &\text{Lorenz gauge:} && \Rightarrow \quad k_\mu A^{\mu\nu} = 0 && (5.7)\\ &\text{Symmetry:} && \Rightarrow \quad A^{\mu\nu} = A^{\nu\mu} && (5.8) \end{aligned}$$

The first equation implies that the wave moves with phase speed $v = 1$:

$$\begin{aligned} 0 = k^\alpha k_\alpha = \eta^{\alpha\beta} k_\alpha k_\beta = \eta^{tt}(-\omega)^2 + \eta^{xx}(k_x)^2 + \eta^{yy}(k_y)^2 + \eta^{zz}(k_z)^2 \\ \Rightarrow \quad 0 = -\omega^2 + k^2 \quad \Rightarrow \quad \omega = k \quad \Rightarrow \quad v = \frac{\omega}{k} = 1 \end{aligned} \tag{5.9}$$

Indeed, the fact that we must have $\omega = k$ means that the waves' group velocity$v_g = d\omega/dk = 1$ as well. So we see that (like electromagnetic waves) gravitational waves in a vacuum are dispersionless and move with the maximum speed allowed by special relativity.

However, just because we have a plane wave solution to the weak-field Einstein equation in empty space does not necessarily mean that we have an actual gravitational wave. In this case, a wavy metric perturbation may describe a physical wavelike distortion of spacetime, but it may just as easily describe a wavy coordinate system on top of a flat spacetime. How can we tell which is which?

There are several ways to do this. One common method is to use our additional gauge freedom (coordinate transformations that satisfy $\Box^2 \xi^\alpha = 0$) to set as many components of $A^{\mu\nu}$ to zero as possible: the remaining components (which we cannot transform away via an allowable coordinate transformation) will then plausibly reflect a wave that is physical. But this method has always left me a bit uncomfortable. Are we absolutely certain that the remaining components represent a physical wave, or just a wavy coordinate system that we cannot erase for some reason with our choice of gauge conditions?

So I am going to present a different method, which has some advantages both in the simplicity of the math and the clarity of its implications (and leave the other approach to a problem). The one true way to tell whether you have a physical gravitational wave or just a wavy coordinate system is to *look at the Riemann tensor*. According to a chart that we constructed a few days ago, we have 21 potentially independent Riemann tensor components (updated for our particular coordinate system):

$$\begin{array}{cccccccc} & \mu\nu \rightarrow & tx & ty & tz & xy & xz & yz \\ \alpha\beta\downarrow & tx & R_{txtx} & R_{txty} & R_{txtz} & R_{txxy} & R_{txxz} & R_{txyz} \\ & ty & & R_{tyty} & R_{tytz} & R_{tyxy} & R_{tyxz} & R_{tyyz} \\ & tz & & & R_{tztz} & R_{tzxy} & R_{tzxz} & R_{tzyz} \\ & xy & & & & R_{xyxy} & R_{xyxz} & R_{xyyz} \\ & xz & & & & & R_{xzxz} & R_{xzyz} \\ & yz & & & & & & R_{yzyz} \end{array} \tag{5.10}$$

subject to the additional constraint that $0 = R_{txyz} + R_{tzxy} + R_{tyzx}$, to leave 20 truly independent components. Now, to make the math easier, let's assume that our plane wave is moving purely in the z direction so that $k_t = -\omega,\ k_x = 0,\ k_y = 0,\ k_z = \omega$ ($k_t = -k_z$ ensures that the condition $k^\alpha k_\alpha = 0$ is satisfied). The Lorenz condition in this case requires that $0 = k_\mu A^{\mu\nu} = -\omega A^{t\nu} + \omega A^{z\nu}$, implying that

$$A^{tt} = A^{zt} \ (= A^{tz}) \tag{5.11}$$

$$A^{tx} = A^{zx} \ (= A^{xt} = A^{xz}) \tag{5.12}$$

$$A^{ty} = A^{zy} \ (= A^{yt} = A^{yz}) \tag{5.13}$$

$$A^{tz} = A^{zz} \ (= A^{zt} = A^{tt} \text{ from above}) \tag{5.14}$$

where the equalities in parentheses follow from the symmetry of $A^{\mu\nu}$. Also note that

$$\partial_\beta \partial_\mu h_{\alpha\nu} = \partial_\beta \partial_\mu (H_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu} H) = k_\beta k_\mu (A_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu} A) \sin k_\sigma x^\sigma \tag{5.15}$$

where $A \equiv \eta_{\mu\nu} A^{\mu\nu} = -A^{tt} + A^{xx} + A^{yy} + A^{zz} = A^{xx} + A^{yy}$ because the Lorenz condition requires that $A^{tt} = A^{zz}$ (see equations 5.11 and 5.14). Finally, note that

$$A_{\alpha\nu} = \eta_{\alpha\beta}\eta_{\nu\mu} A^{\beta\mu} = \begin{cases} -A^{\alpha\nu} & \text{if } \textit{either } \alpha = t \text{ or } \nu = t \text{ but not both} \\ +A^{\alpha\nu} & \text{otherwise} \end{cases} \tag{5.16}$$

We are now ready to evaluate components of the Riemann tensor, which in this weak-field limit is

$$\begin{aligned}R_{\alpha\beta\mu\nu} &= \tfrac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}) \\ &= -\tfrac{1}{2}(k_\beta k_\mu[A_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu}A] + k_\alpha k_\nu[A_{\beta\mu} - \tfrac{1}{2}\eta_{\beta\mu}A] \\ &\qquad - k_\alpha k_\mu[A_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}A] - k_\beta k_\nu[A_{\alpha\mu} - \tfrac{1}{2}\eta_{\alpha\mu}A])\sin k_\sigma x^\sigma \end{aligned} \tag{5.17}$$

Now let's start calculating components of this tensor. For example:

$$\begin{aligned}R_{txtx} &= -\tfrac{1}{2}(k_x k_t[A_{xt} - \tfrac{1}{2}\eta_{xt}A] + k_t k_x[A_{tx} - \tfrac{1}{2}\eta_{tx}A] \\ &\qquad - k_t k_t[A_{xx} - \tfrac{1}{2}\eta_{xx}A] - k_x k_x[A_{tt} - \tfrac{1}{2}\eta_{tt}A])\sin k_\sigma x^\sigma \\ &= -\tfrac{1}{2}(0 + 0 - \omega^2[A_{xx} - \tfrac{1}{2}(A_{xx} + A_{yy})] - 0)\sin k_\sigma x^\sigma \\ &= +\tfrac{1}{4}\omega^2(A_{xx} - A_{yy})\sin k_\sigma x^\sigma \end{aligned} \tag{5.18}$$

This term will be nonzero (indicating a physically curved spacetime, and thus a real gravitational wave) if and only if $A_{xx} - A_{yy}$ is nonzero. Now let's look at

$$\begin{aligned}R_{txty} &= -\tfrac{1}{2}(k_x k_t[A_{ty} - \tfrac{1}{2}\eta_{ty}A] + k_t k_y[A_{xt} - \tfrac{1}{2}\eta_{xt}A] \\ &\qquad - k_t k_t[A_{xy} - \tfrac{1}{2}\eta_{xy}A] - k_x k_y[A_{tt} - \tfrac{1}{2}\eta_{tt}A])\sin k_\sigma x^\sigma \\ &= -\tfrac{1}{2}(0 + 0 - \omega^2 A_{xy} - 0)\sin k_\sigma x^\sigma = +\tfrac{1}{2}\omega^2 A_{xy}\sin k_\sigma x^\sigma \end{aligned} \tag{5.19}$$

So this term will be nonzero if and only if A_{xy} is nonzero. Similarly,

$$\begin{aligned}R_{txtz} &= -\tfrac{1}{2}(k_x k_t[A_{tz} - \tfrac{1}{2}\eta_{tz}A] + k_t k_z[A_{xt} - \tfrac{1}{2}\eta_{xt}A] \\ &\qquad - k_t k_t[A_{xz} - \tfrac{1}{2}\eta_{xz}A] - k_x k_z[A_{tt} - \tfrac{1}{2}\eta_{tt}A])\sin k_\sigma x^\sigma \\ &= -\tfrac{1}{2}(0 - \omega^2 A_{xt} - \omega^2 A_{xz} - 0)\sin k_\sigma x^\sigma \\ &= +\tfrac{1}{2}\omega^2(A_{xt} + A_{xz})\sin k_\sigma x^\sigma = 0 \end{aligned} \tag{5.20}$$

because the Lorenz condition requires that $A^{xt} = A^{xz}$, so $A_{xt} = -A_{xz}$, meaning that this term is identically zero, no matter what A^{xt} is. Similarly, $R_{tytz} = 0$ no matter what A^{yt} is. Now let's look at

$$\begin{aligned}R_{txxy} &= -\tfrac{1}{2}(k_x k_x[A_{ty} - \tfrac{1}{2}\eta_{ty}A] + k_t k_y[A_{xx} - \tfrac{1}{2}\eta_{xx}A] \\ &\qquad - k_t k_x[A_{xy} - \tfrac{1}{2}\eta_{xy}A] - k_x k_y[A_{tx} - \tfrac{1}{2}\eta_{tx}A])\sin k_\sigma x^\sigma \\ &= -\tfrac{1}{2}(0 + 0 - 0 - 0)\sin k_\sigma x^\sigma = 0 \end{aligned} \tag{5.21}$$

This one is identically zero because we need at least two t and/or z indices for at least one pair of k's to be nonzero. The same reasoning also eliminates $R_{tyxy}, R_{xyxy}, R_{xyxz}$, and R_{xyyz}. Finally, consider

$$\begin{aligned}R_{txyz} &= -\tfrac{1}{2}(k_x k_y[A_{tz} - \tfrac{1}{2}\eta_{tz}A] + k_t k_z[A_{xy} - \tfrac{1}{2}\eta_{xy}A] \\ &\qquad - k_t k_y[A_{xz} - \tfrac{1}{2}\eta_{xz}A] - k_x k_z[A_{ty} - \tfrac{1}{2}\eta_{ty}A])\sin k_\sigma x^\sigma \\ &= -\tfrac{1}{2}(0 - \omega^2 A_{xy} - 0 - 0)\sin k_\sigma x^\sigma = \tfrac{1}{2}\omega^2 A_{xy}\sin k_\sigma x^\sigma \end{aligned} \tag{5.22}$$

which again tells us that the gravitational wave is real when $A_{xy} \neq 0$.

In a similar way, you can easily analyze the remaining terms yourself. Here is a complete list of the 21 Riemann tensor terms:

$$\begin{array}{cccccccc} & \mu\nu\rightarrow & tx & ty & tz & xy & xz & yz \\ \alpha\beta\downarrow & tx & R_{txtx}=a & R_{txty}=b & R_{txtz}=0 & R_{txxy}=0 & R_{txxz}=a & R_{txyz}=b \\ & ty & & R_{tyty}=-a & R_{tytz}=0 & R_{tyxy}=0 & R_{tyxz}=b & R_{tyyz}=a \\ & tz & & & R_{tztz}=0 & R_{tzxy}=0 & R_{tzxz}=0 & R_{tzyz}=0 \\ & xy & & & & R_{xyxy}=0 & R_{xyxz}=0 & R_{xyyz}=0 \\ & xz & & & & & R_{xzxz}=a & R_{xzyz}=b \\ & yz & & & & & & R_{yzyz}=-a \end{array} \tag{5.23}$$

where $a \equiv \frac{1}{4}\omega^2(A_{xx} - A_{yy})\sin k_\sigma x^\sigma$ and $b \equiv \frac{1}{2}\omega^2 A_{xy}\sin k_\sigma x^\sigma$. You can also see that the symmetry condition $0 = R_{txyz} + R_{tzxy} + R_{tyzx}$ imposes no additional constraints.

The point of this is that for a wave moving in the z direction, the only parts of $A^{\mu\nu}$ that actually matter physically are two values: the value of $A^{xx} - A^{yy}$ and the value of $A^{xy} = A^{yx}$. All of the other components have no physical consequences, meaning that we should be able to find a coordinate transformation that sets them to zero. Also, since only the *difference* between A^{xx} and A^{yy} matters, we should be able to go to a coordinate system where we subtract the remaining trace $A = A^{xx} + A^{yy}$ of the metric from each component:

$$A^{xx}_{\text{new}} = A^{xx} - \tfrac{1}{2}A = A^{xx} - \tfrac{1}{2}(A^{xx} + A^{yy}) = \tfrac{1}{2}(A^{xx} - A^{yy}) \tag{5.24a}$$
$$A^{yy}_{\text{new}} = A^{yy} - \tfrac{1}{2}A = A^{yy} - \tfrac{1}{2}(A^{xx} + A^{yy}) = -\tfrac{1}{2}(A^{xx} - A^{yy}) \tag{5.24b}$$

This will make the A-matrix traceless ($A_{new} = 0$) without affecting the difference between these components: $A^{xx}_{\text{new}} - A^{yy}_{\text{new}} = \frac{1}{2}(A^{xx} - A^{yy}) + \frac{1}{2}(A^{xx} - A^{yy}) = A^{xx} - A^{yy}$. We call the gauge where waves traveling in the $+z$ direction have the metric perturbation

$$H^{\mu\nu}_{TT} = \left(A_+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + A_\times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \cos k_\sigma x^\sigma \tag{5.25}$$

the **transverse traceless gauge** for that wave (because the only nonzero spatial components are transverse to the wave's direction of motion and the matrix also has zero trace). We see that the general physically relevant wave is a linear combination (with small but otherwise arbitrary values of the coefficents A_+ and $A_\times$) of two independent types of waves, which we call **polarizations** of the gravitational wave (in analogy to the two linear polarizations of electromagentic waves). We call these polarization states "upright" and "diagonal," or "plus" and "cross" (for reasons that will become clear shortly).

Note also that in transverse-traceless gauge, there is no distinction between $H^{\mu\nu}$ and $h^{\mu\nu}$:

$$h^{\mu\nu}_{TT} = H^{\mu\nu}_{TT} - \tfrac{1}{2}\eta^{\mu\nu} H_{TT} = H^{\mu\nu}_{TT} \tag{5.26}$$

This is one of the reasons it is convenient to set the trace to zero.

To determine the physical effects of the wave, consider a particle at rest ($u^x = u^y = u^z = 0$). The geodesic equation becomes

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu = -\Gamma^\alpha_{tt} u^t u^t = -\tfrac{1}{2}\eta^{\alpha\beta}(\partial_t h^{TT}_{t\beta} + \partial_t h^{TT}_{\beta t} - \partial_\beta h^{TT}_{tt}) u^t u^t = 0\,! \tag{5.27}$$

because the metric perturbation in transverse-traceless gauge has no nonzero components with a t in one or more index. This *looks* like it says that the wave has no physical effect on the particle after all! But this is *not* what the equation is really saying. Just because the coordinates have the names t, x, y, z does not make them cartesian coordinates. What the equation is actually saying is that in this case, the coordinates are simply *comoving* with a free particle so that the coordinates of that particle remain fixed.

To see that this does not imply that the wave has no effect, we need (as always!) to go back to the metric to see what our physical coordinates actually *mean*. Rather than one particle, consider a set of particles in the xy plane that (before the wave comes by) are arranged in a ring of radius R that a purely "uprightly" or "plus" polarized gravitational plane wave ($A_\times = 0$) moves in the $+z$ direction through this ring. The displacements $\Delta x = R\cos\theta$ and $\Delta y = R\sin\theta$ will be fixed in the transverse-traceless gauge, as we have seen. But their distances Δs from the center at a given instant of time t ($\Delta t = 0$ from the center) on the xy plane ($\Delta z = 0$) are *not* fixed: according to the metric equation, we have

$$\begin{aligned} \Delta s^2 &= (\eta_{xx} + h^{TT}_{xx})\Delta x^2 + (\eta_{yy} + h^{TT}_{yy})\Delta x^2 \\ &= (1 + A_+)R^2\cos^2\theta\cos^2\omega t + (1 - A_+)R^2\sin^2\theta\cos^2\omega t \\ &= R^2[1 + A_+(\cos^2\theta - \sin^2\theta)]\cos^2\omega t = R^2(1 + A_+\cos 2\theta)\cos^2\omega t \\ \Rightarrow \quad \Delta s &= R(1 + A_+\cos 2\theta)^{1/2}\cos\omega t \approx R(1 + \tfrac{1}{2}A_+\cos 2\theta)\cos\omega t \end{aligned} \tag{5.28}$$

where in the last step I have used the binomial approximation, since $A_+ \ll 1$. Similarly, you can show that for a "diagonally" or "cross" polarized wave, we have

$$\Delta s \approx R(1 + \tfrac{1}{2}A_\times \sin 2\theta)\cos\omega t \tag{5.29}$$

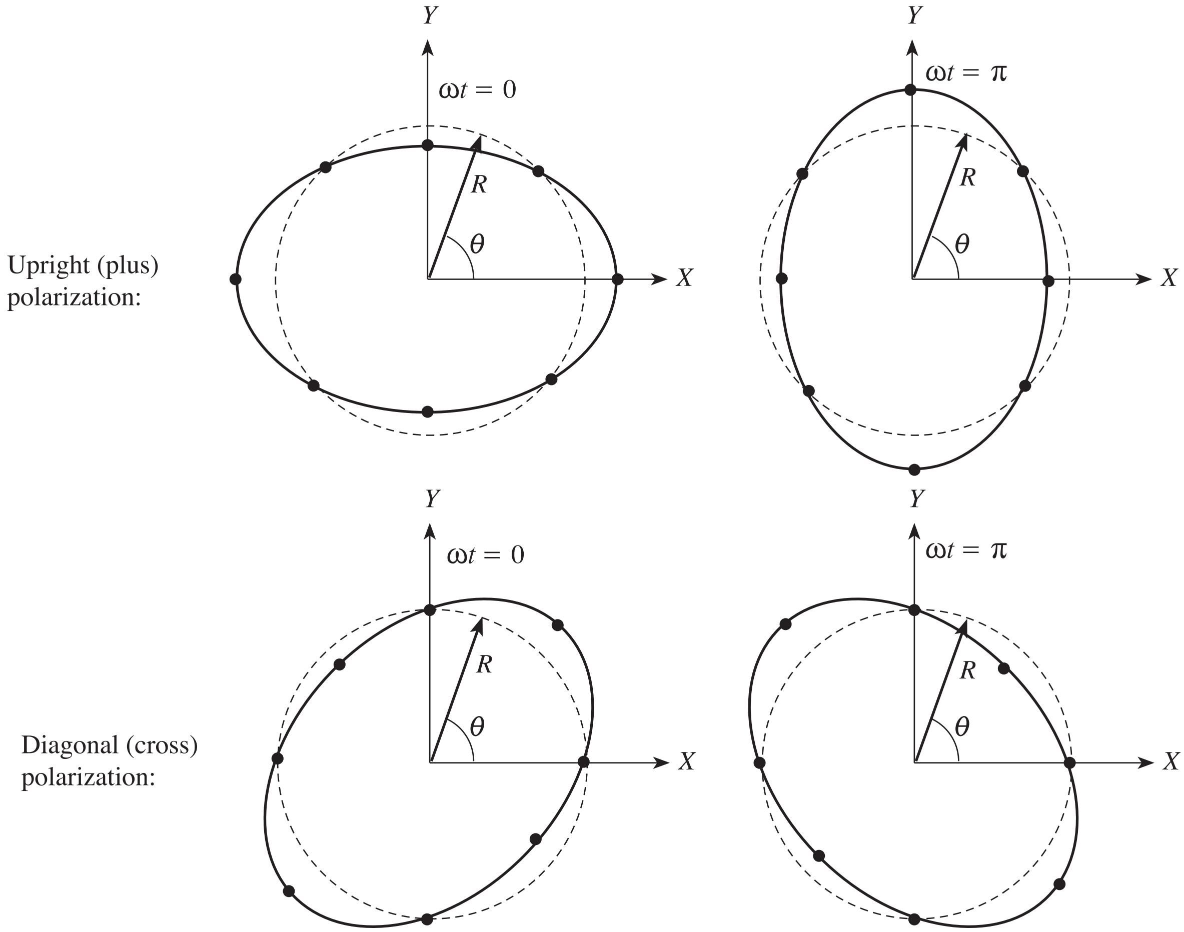

Figure 1: This diagram shows how a gravitational wave moving in the $+z$ direction deforms a ring of free particles floating in space. The X and Y coordinates here are coordinates of a LIF, where coordinate displacements are the same as distance displacements.

We see that the distance that each particle is from the center oscillates as the wave passes and is proportional to the displacement R that the particle is from the center when no wave is present. The wave deforms the ring as shown in figure 1.

The deformation for the "upright" or "plus" polarization has the long axis of the ellipse oscillating between the X and Y axes (in LIF coordinates that correspond to the actual distances), while that for the "diagonal" or "cross" polarization is the same except rotated 45°. You can now see why the subscripts on A_+ and $A_\times$ make sense. This is also why the international symbol for dangerous gravitational radiation is as shown in figure 2. (OK, sorry, that is a geeky joke.)

"Dangerous gravitational radiation" is actually practically a contradiction in terms. We will see that a wide variety of astrophysical sources produce gravitational waves with amplitudes A_+ and/or $A_\times$ on the order of 10^{-18} or smaller at the earth. This means that free particles separated by $R \approx 1 \times 10^6\,\mathrm{m} = 1000\,\mathrm{km}$ would oscillate back and forth with amplitudes on the order of magnitude of 10^{-12} m (about a hundredth of the size of an atom) even for waves with the largest likely amplitudes. The implication is that gravitational waves from astrophysical sources are extremely hard to even detect, much less harm someone.

But humans have in fact detected gravitational waves, as you know! The basic design of almost any detector involves a set of floating masses which have been extraordinarily well-isolated from their surroundings, and some kind of laser-ranging system that accurately measures the physical distance between the floating objects using interferometry. (One might think that this would be impossible with visible-light lasers, since the displacement of each mass when the wave comes by will be such a tiny fraction of the wavelength of light. But if the laser light is bright enough, one can detect a shift in the interference pattern from complete destructive interference to just less than complete destructive interference, even if the shift is only a tiny fraction of the wavelength of light.

Figure 2: International warning symbol for dangerous gravitational radiation.

My personal research has to do with LISA, a space-based gravitational wave detector that will be flown by the ESA, probably in about 2034. The floating masses in LISA's case would be "proof masses" isolated in the centers of three satellites arranged in an equilateral triangle about 5 million km on a side. The array would orbit the sun about 20° behind the Earth in its orbit. The recent LISA Pathfinder mission tested the basic technology, which performed even better than expected.

In spite of the fact that LISA's arms are much longer than those of the ground-based detectors LIGO and VIRGO, the sensitivity would be about the same, because the interferometer must be much more robust (there is no ability to tinker with it once it is launched!) and also the power available to the lasers would be much smaller (particularly considering the distances involved). The advantage of a space-based detector is that the floating masses will be isolated from seismic noise, which is the main limit on LIGO's noise curve at low frequencies. While LIGO is sensitive to waves in an essentially audio frequency range of about 10 Hz to 1000 Hz, LISA will be sensitive to waves in the 10^{-4} Hz to 10^{-2} Hz (periods of hours to minutes). There are *many* such sources of gravitational waves in that frequency range, from close binaries in our own galaxy to coalescing supermassive black holes (which would be bright enough to be registered anywhere in the universe). Indeed the limit on LISA's noise curve at the low-frequency end is that there are so many binary star systems in our galaxy with periods of hours or longer that isolated sources will be lost in the "crowd noise."

5.2.1 Exercise: Other Riemann Components

Calculate one or more of the Riemann tensor components in equations 5.23 to verify the results quoted there.

5.2.2 Exercise: Cross Polarization

Verify equation 5.29 for the deformation of a ring of particles due to a cross-polarized wave. (You will need a double-angle trig identity that you can easily look up online.)

5.3 Generating Gravitational Waves

One can very crudely estimate the maximum strength of gravitational waves generated by an astrophysical source as follows. At the source (say, for example, a pair of coalescing black holes), the metric perturbations right at the source will be at most of order of magnitude 1 (note that $g_{tt} = -(1 - 2GM/r) = -1 + 1$ at $r = 2GM$, which is the radius of a black hole's event horizon). The coalescing black holes that LIGO detected in 2015 had a total mass of about 60 solar masses, for a total $GM \approx 10^5$m. Since gravitational waves, like light waves, fall off in amplitude as $1/r$, such waves would have the observed amplitude of $A \approx 10^{-21}$ at a distance of about 10^{26} m $\approx 10^{10}$ ly. Since the actual estimate of that source's distance was 1.3 Gy, our crude estimate is actually not bad at all.

This means that if our sun were to become a black hole and coalesce with another solar-mass black hole, for a total $GM \approx 3000\,\mathrm{m}$, the gravitational waves at the earth at a distance of $r \approx 1.5 \times 10^{11}$ m would have an amplitude on the order of magnitude of $1/(1.5 \times 10^{11}/3000) \approx 2 \times 10^{-8}$. While this would probably blow LIGO's gaskets, I don't think that you would even feel your body want to stretch and shrink by about 200 atom-widths.

Significantly more accurate guesses about the gravitational waves produced by coalescing black holes require detailed computer models, and predicting how often such events occur is also very difficult. However, simple binary star systems are *known and steady* sources of gravitational waves, and one can make realistic *analytical* calculations for the waves emitted by such sources. In what follows, I will describe how to calculate the gravitational waves emitted by a system with two or more moving parts using the *small-slow-weak* approximation, where we assume that

1. The source is *small* compared to both the wave's wavelength and the distance to the observer.

2. The source is *weak* in that $|h_{\mu\nu}| \ll 1$ even very near the source. (This will be reasonable for most astrophysical sources of gravitational waves except for coalescing black holes or neutron stars.)

3. The source is *slow* in that parts of the source move with speeds $v \ll 1$. (This again will be true for most astrophysical sources other than coalescing black holes or neutron stars.)

Let's see how these approximations can help. The weak-field limitation allows us to us the weak-field Einstein equation even to describe the source. We seek to solve

$$\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu} \quad \text{subject to the Lorenz condition } \partial_\mu H^{\mu\nu} = 0 \tag{5.30}$$

We saw in the last session that solutions to this equation are (by analogy to solutions of the corresponding electrostatic equation $\Box^2\phi = -4\pi k\rho$) are

$$H^{\mu\nu}(t,\vec{R}) = 4G\int_{\text{src}} \frac{T^{\mu\nu}(t-s,\vec{r})\,dV}{s} \quad \text{where } s \equiv |\vec{R}-\vec{r}\,| \tag{5.31}$$

and the integral is a volume integral over positions $\vec{r}$ in the source. Now, if the source is also *small* compared to $R \equiv |\vec{R}|$, then $s \approx R$. If the source is also small compared to a wavelength of the wave, then the retarded time $t - s \approx t - R$ for all points on the source. In this case, the solution becomes the simpler function

$$H^{\mu\nu}(t,\vec{R}) = \left[\frac{4G}{R}\int_{\text{src}} T^{\mu\nu}\,dV\right]_{\text{at } t-R} \tag{5.32}$$

From now on, let's *assume* that all integrals over the source are calculated at the retarded time $t - R$ (so that we don't have to write this over and over).

I am now going to give you an overview of where we are going so that you do not get distracted by the mathematics required to get there. We will see that if the source's center of mass is at rest in our coordinate system, then $H^{tt} = 4GM/R =$ constant, and $H^{ti} = H^{it} = 0$ (where, again, I am using the convention that Latin-letter indices range only over the *spatial* component index values). Therefore, the only potentially "waving" components of the trace-reversed metric perturbation $H^{\mu\nu}$ are the spatial components H^{ij}. We will then show that the divergence theorem in conjunction with conservation of energy and momentum in the source together imply that

$$\int_{\text{src}} T^{ij}\,dV = \frac{1}{2}\frac{d^2}{dt^2}\int_{\text{src}} T^{tt}x^i x^j\,dV = \frac{1}{2}\frac{d^2}{dt^2}\int_{\text{src}} \rho x^i x^j\,dV \equiv \frac{1}{2}\ddot{I}^{ij} \tag{5.33}$$

where $I^{ij} \equiv \int \rho x^i x^j\,dV$ is the source's **quadrupole moment tensor** (note that it is a tensor only with regard to rotations and displacements of the spatial coordinates). It turns out to be useful (for a number of reasons) to use expressions involving the traceless **reduced quadrupole moment tensor**

$$\rlap{-}I^{ij} \equiv \int_{\text{src}} \rho(x^i x^j - \tfrac{1}{3}\eta^{ij}r^2)\,dV \quad \text{where } r^2 \equiv x^2 + y^2 + z^2 \tag{5.34}$$

instead. This matrix is automatically traceless. It is also the tensor that one would use to expand the Newtonian gravitational potential Φ at some large distance R from a compact and static but asymmetrical source whose center of mass is a the origin:

$$\Phi = -\frac{GM}{R} - \frac{3\rlap{-}I_{ij}}{2R^3}\left(\frac{X^i}{R}\right)\left(\frac{X^j}{R}\right) + \cdots \tag{5.35}$$

where X^i is a component of the radius vector $\vec{R}$ from the source to the observer. Therefore, the reduced quadrupole moment tensor expresses the leading component of the field's asphericity.

So the physically significant transverse-traceless components of the gravitational waves from a *small-slow-weak* source (assuming that the observer is at a point in the $+z$ direction relative to the source) are

$$H^{xx}_{TT} = \tfrac{1}{2}(H^{xx} - H^{yy}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I}^{xx} - \ddot{I}^{yy}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I\!\!\!-}^{xx} - \ddot{I\!\!\!-}^{yy}) \equiv \frac{2G}{R}\ddot{I\!\!\!-}^{xx}_{TT} \tag{5.36a}$$

$$H^{yy}_{TT} = \tfrac{1}{2}(H^{yy} - H^{xx}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I}^{yy} - \ddot{I}^{xx}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I\!\!\!-}^{yy} - \ddot{I\!\!\!-}^{xx}) \equiv \frac{2G}{R}\ddot{I\!\!\!-}^{yy}_{TT} \tag{5.36b}$$

$$H^{xy}_{TT} = \frac{2G}{R}\ddot{I}^{xy} = \frac{2G}{R}\ddot{I\!\!\!-}^{xy} \equiv \frac{2G}{R}\ddot{I\!\!\!-}^{xy}_{TT} \tag{5.36c}$$

Note that $\ddot{I}^{xx} - \ddot{I}^{yy} = \ddot{I\!\!\!-}^{xx} - \ddot{I\!\!\!-}^{yy}$ because the extra term $-\frac{d^2}{dt^2}[\int \rho r^2 dV]$ that appears in the reduced quadrupole moment tensor terms cancels out of the difference. Equations 5.36 therefore define the "transverse-traceless" components of the reduced quadrupole moment tensor for waves traveling in the $+z$ direction. In many cases, we will be able to orient our coordinate system so that waves for a given observer are moving in this direction.

Note that a spherically symmetric source has zero quadrupole moment tensor, and therefore will not radiate gravitational waves, even if it is expanding or contracting. According to **Birchoff's theorem**, this statement is actually true no matter how strong the gravitational fields are, how relativistic the source is, and how close we are to the source. For a proof of this theorem, see chapter 23 in my book.[1]

Now that we see the big picture, let's dig into the mathematics. First consider the metric perturbation components $H^{t\mu}$. According to the definitions of the stress-energy components, in a LIF, we have

$$T^{tt} = \text{density of energy } = \rho \tag{5.37a}$$

$$T^{it} = T^{ti} = \text{density of } i\text{-momentum} \tag{5.37b}$$

This should also be true (to the level of our approximations) in the "nearly cartesian" we use in the weak-field limit (since $T^{\mu\nu}$ is already of the order of $h^{\mu\nu}$, corrections will be of order $|h^{\mu\nu}|^2$ and so are negligible). Therefore, we have

$$\int_{\text{src}} T^{\,tt}\, dV = \int_{\text{src}} \rho\, dV = M, \quad \text{and} \int_{\text{src}} T^{\,ti}\, dV = \int_{\text{src}} T^{\,it}\, dV = P^i \tag{5.38}$$

where M is the source's total mass-energy and $\vec{P}$ is its total momentum. But if we anchor our coordinates to the source's center of mass, then $\vec{P} = 0$. If our source is also "small," then equation 5.32 tells us that

$$H^{tt} = \frac{4G}{R}\int_{\text{src}} T^{\,tt} dV = \frac{4GM}{R} \quad \text{and} \quad H^{ti} = \frac{4G}{R}\int_{\text{src}} T^{\,ti} dV = 0 = H^{it} \tag{5.39}$$

as I claimed earlier.

Now let's consider the metric perturbation components H^{ij}. In a LIF, conservation of energy requires that $\partial_\mu T^{\mu\nu} = 0$. Again, this should still be true to our level of approximation in our "nearly cartesian" coordinates. If we break this up into time and space parts, we have

$$0 = \partial_t T^{\,t\nu} + \partial_i T^{\,i\nu} \quad \Rightarrow \quad \partial_t T^{\,t\nu} = -\partial_i T^{\,i\nu} \tag{5.40}$$

Now, note that in the quantity $T^{\,tt}x^i x^j$ that appears in equation 5.33, x^i and x^j are the components of the position of a volume element in the source. Therefore, they are independent of time: though $T^{\,tt}$ inside the volume element may vary with time, the position of the volume element itself will not. Therefore, we can do the following steps of calculation:

$$\partial_t\partial_t(T^{\,tt}x^i x^j) = (\partial_t\partial_t T^{\,tt})x^i x^j \tag{5.41a}$$

$$= -(\partial_t\partial_m T^{\,mt})x^i x^j \tag{5.41b}$$

$$= -(\partial_m\partial_t T^{\,tm})x^i x^j \tag{5.41c}$$

$$= +(\partial_m\partial_n T^{\,nm})x^i x^j \tag{5.41d}$$

$$= \partial_m\partial_n(T^{\,mn}x^i x^j) - 2\partial_n(T^{\,ni}x^j + T^{\,nj}x^i) + 2T^{\,ij} \tag{5.41e}$$

The last step follows from the product rule, though the argument is a bit involved. Start by applying the product rule to the first term on the right of the last equation:

$$\begin{aligned}
\partial_m \partial_n (T^{mn} x^i x^j) &= \partial_m [\partial_n (T^{mn} x^i x^j)] && (5.42a)\\
&= \partial_m [(\partial_n T^{mn}) x^i x^j + T^{mn} \delta^i_n x^j + T^{mn} x^i \delta^j_n] && (5.42b)\\
&= \partial_m [(\partial_n T^{mn}) x^i x^j + T^{mi} x^j + T^{mj} x^i] && (5.42c)\\
&= (\partial_m \partial_n T^{mn}) x^i x^j + (\partial_n T^{mn}) \delta^i_m x^j + (\partial_n T^{mn}) x^i \delta^j_m + \partial_m (T^{mi} x^j + T^{mj} x^i) && (5.42d)\\
&= (\partial_m \partial_n T^{mn}) x^i x^j + (\partial_n T^{in}) x^j + (\partial_n T^{jn}) x^i + \partial_m (T^{mi} x^j + T^{mj} x^i) && (5.42e)\\
&= \partial_m \partial_n (T^{mn}) x^i x^j + \partial_n (T^{in} x^j) - T^{ij} + \partial_n (T^{jn} x^i) - T^{ji} + \partial_m (T^{mi} x^j + T^{mj} x^i) && (5.42f)\\
&= \partial_m \partial_n (T^{mn}) x^i x^j + 2\partial_n (T^{in} x^j + T^{jn} x^i) - 2T^{ij} && (5.42g)
\end{aligned}$$

Solving for the first term on the right of this equation then gives us

$$\partial_m \partial_n (T^{mn}) x^i x^j = \partial_m \partial_n (T^{mn} x^i x^j) - 2\partial_n (T^{in} x^j + T^{jn} x^i) + 2T^{ij} \tag{5.43}$$

which is precisely the step from equation 5.41d to the next equation 5.41e.

Whew! After all that math, we still don't seem to have something very pretty. But the divergence theorem says that for an arbitrary 3-vector field $\vec{F}(x, y, z)$,

$$\int_V \vec{\nabla} \cdot \vec{F}\, dV = \oint_S \vec{F} \cdot d\vec{A} \quad \text{or} \quad \int_V \partial_i F^i\, dV = \oint_S F^i\, dA_i \quad \text{in index notation} \tag{5.44}$$

The same is true for the integral of an arbitrary 3-tensor field:

$$\int_V \partial_i F^{ij}\, dV = \oint_S F^{ij}\, dA_i \tag{5.45}$$

(You can think of the jth column of the matrix F^{ij} as being an independent 3-vector field.) Now, imagine integrating both sides of equation 5.41e (repeated here)

$$\partial_t \partial_t (T^{tt} x^i x^j) = \partial_m \partial_n (T^{mn} x^i x^j) - 2\partial_n (T^{ni} x^j + T^{nj} x^i) + 2T^{ij} \tag{5.46}$$

over a volume large enough to completely enclose the source, so $T^{ij} = 0$ on the volume's surface. The divergence theorem then implies that the first two terms on the right integrate to zero, leaving us with

$$\int_V \partial_t \partial_t (T^{tt} x^i x^j)\, dV = \int_V 2T^{ij}\, dV \tag{5.47}$$

We can then pull the double partial derivative outside the left integral (where it becomes an ordinary double time derivative, since integral as a whole can only depend on time), divide both sides by 2, and substitute in $I^{ij} \equiv \int_V \rho x^i x^j\, dV = \int_V T^{tt} x^i x^j\, dV$ to get

$$\int_{\text{src}} T^{ij}\, dV = \frac{1}{2} \frac{d^2}{dt^2} \int_{\text{src}} \rho x^i x^j\, dV = \frac{1}{2} \ddot{I}^{ij} \tag{5.48}$$

which is precisely equation 5.33. This was a long journey, but the result is worth it!

5.3.1 Exercise: Math Check.

Go through each of the steps in equations 5.41 and 5.42 to make sure that you understand exactly what I did in each step.

5.4 Gravitational Wave Energy.

Now that we have calculated the gravitational waves that a source might radiate, we would very much like to know how much energy they carry away from the source. Now, as we have discussed before, the concept of energy conservation in general relativity is a difficult and potentially contentious topic. However, in the weak-field limit, we are essentially using flat spacetime as a background, where the kind of integrals that we need to do to make global (as opposed to local) statements about conservation of energy are possible.

In this particular case, we can employ a generally accepted trick that satisfies one's intuition about how gravitational waves should conserve energy and which allows us to calculate an effective energy carried by the waves in this limit.

The Einstein equation to first order in the metric perturbation $h_{\mu\nu}$ (and in the coordinates defined by the Lorenz gauge condition) tells us that

$$-2G^{(1)}_{\mu\nu} = \Box^2 H_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{5.49}$$

where $G^{(1)}_{\mu\nu}$ is the Einstein tensor evaluated to first order in $h_{\mu\nu}$ and $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ as we have defined before. This equation describes how a gravitational field is created by the density of non-gravitational energy and momentum expressed by the stress-energy tensor $T^{\mu\nu}$.

Now the full Einstein equation also include feedback effects that describe how the gravitational field affects itself: these are expressed by the fact that the Einstein tensor is nonlinear in the metric (contains terms where metric components are multiplied together). People sometimes loosely (and arguably inaccurately) interpret these internal feedback effects in the strong-field limit as telling us that the gravitational field has an energy that is also the source of a gravitational field. But however one interprets these nonlinearities, they do not appear in the weak-field approximation, because we are dropping terms of order $|h_{\mu\nu}|^2$ (indeed, that is the *point* of the weak-field limit!).

The commonly accepted trick is to expand the left side of the weak-field Einstein equation to *second* order in the metric:

$$-2G^{(1)}_{\mu\nu} - 2G^{(2)}_{\mu\nu} = \Box^2 H_{\mu\nu} - 2G^{(2)}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{5.50}$$

(We can get away with only doing this to the left side, because on the right side, the second-order metric terms $h_{ij}h_{mn}$ that in transverse-traceless gauge might actually appear on the right multiply pressure and momentum flow terms that are much smaller than $T^{tt} = \rho$ in the "slow" limit where the fluid is nonrelativistic. Since T^{tt} is already of order $h_{\mu\nu}$, the second-order terms on the right remain negligible.) Now remember that in the weak-field limit, we are pretending that spacetime is flat and that gravity is completely described by a tensor field $H_{\mu\nu}$ that sits on top of that flat spacetime. Moving the $2G^{(-2)}_{\mu\nu}$ term to the other side yields

$$\Box^2 H_{\mu\nu} = -16\pi G T_{\mu\nu} + 2G^{(2)}_{\mu\nu} = -16\pi G(T_{\mu\nu} + T^{GW}_{\mu\nu}) \quad \text{where} \quad T^{GW}_{\mu\nu} \equiv \frac{G^{(2)}_{\mu\nu}}{8\pi G} \tag{5.51}$$

In this way of interpreting the equation, $T^{GW}_{\mu\nu}$ is acting along with the non-gravitational stress-energy $T_{\mu\nu}$to create the gravitational field $H_{\mu\nu}$, so it is acting like a stress-energy of the gravitational field. Moreover, since $\partial_\mu H^{\mu\nu} = 0$ in the Lorenz gauge we are working in, so if we raise indices on both sides of the equation above and then take the divergence of both sides, we get

$$\partial_\mu(T^{\mu\nu} + T^{\mu\nu}_{GW}) = 0 \tag{5.52}$$

which expresses local conservation of the *sum* of matter-energy and gravitational field energy, and because we are pretending we are in flat spacetime, we can integrate this to express a global conservation law (gravitational wave energy that crosses the surface of a distant surface surrounding the source comes at the expense of energy in the source).

The only problem is that the quantity $T^{\mu\nu}_{GW}$ is a tensor only with regard to Lorentz transformations, not general coordinate transformations. It is not even invariant with regard to our gauge transformations. It only begins to make sense if we average over several wavelengths of the the gravitational wave because it happens that the terms that ruin the invariance average to zero. So as long as we are willing to accept these limitations, modifying the definition of the gravitational stress-energy to read

$$T^{GW}_{\mu\nu} \equiv \frac{\langle G^{(2)}_{\mu\nu}\rangle}{8\pi G} \tag{5.53}$$

where the $\langle\ \rangle$ brackets indicate an average over several wavelengths, then we have something that we can meaningfully treat as representing the energy of a gravitational wave.

So our next task is to actually evaluate this quantity for a gravitational wave in transverse-traceless coordinates for a plus-polarized gravitational plane wave moving in the $+z$ direction. Let's define the following short-hand expressions for quantities involved in the wave:

$$h_+(t,z) \equiv A_+\cos(\omega t - \omega z) = h^{TT}_{xx} = -h^{TT}_{yy} \tag{5.54a}$$

$$\dot{h}_+ \equiv \partial_t h_+ = -\partial_z h_+ = -A_+\omega\sin(\omega t - \omega z) \tag{5.54b}$$

$$\ddot{h}_+ \equiv \partial_t\partial_t h_+ = \partial_z\partial_z h_+ = -\partial_t\partial_z h_+ = -\partial_z\partial_t h_+ = -\omega^2 h_+ \tag{5.54c}$$

For this perturbation, the metric is completely diagonal so we can use the Diagonal Metric Worksheet (with $A = 1, B = 1 + h_+, C = 1 - h_+, D = 1$) to evaluate the Ricci tensor. Note that

$$B_0 = C_3 = -C_0 = -B_3 = \dot{h}_+ \tag{5.55a}$$

$$B_{00} = B_{33} = C_{03} = -B_{03} = -C_{00} = -C_{33} = \ddot{h}_+ \tag{5.55b}$$

and all other derivatives are nonzero. So, for example, the only nonzero terms in the Diagonal Metric Worksheet's expansion for $R_{00} = R_{tt}$ are:

$$\begin{aligned} R_{tt} &= -\tfrac{1}{2B}B_{00} - \tfrac{1}{2C}C_{00} + \tfrac{1}{4B^2}B_0^2 + \tfrac{1}{4C^2}C_0^2 \\ &= -\frac{\ddot{h}_+}{2(1+h_+)} - \frac{-\ddot{h}_+}{2(1-h_+)} + \frac{\dot{h}_+^2}{4(1+h_+)^2} + \frac{\dot{h}_+^2}{4(1-h_+)^2} \end{aligned} \tag{5.56}$$

Now, we are only trying to keep through order h_+^2, so we can use the binomial approximation to rewrite the denominators in the first two terms, because we only need values representing the denominators to be accurate to order h_+. For the second two terms, the numerators are already second order in h_+, so the denominators are simply 4 to this order. This leaves us with

$$R_{tt} = -\tfrac{1}{2}\ddot{h}_+(1 - h_+) + \tfrac{1}{2}\ddot{h}_+(1 + h_+) + \tfrac{1}{2}\dot{h}_+^2 = \ddot{h}_+ h_+ + \tfrac{1}{2}\dot{h}_+^2 \tag{5.57}$$

Now we average over several wavelengths:

$$\begin{aligned} \langle R_{tt}\rangle &= \langle \ddot{h}_+ h_+ + \tfrac{1}{2}\dot{h}_+^2\rangle = \langle -A_+^2\omega^2\cos^2\theta + \tfrac{1}{2}A_+^2\omega^2\sin^2\theta\ \rangle \\ &= -\omega^2 A_+^2\langle\cos^2\theta - \sin^2\theta - \tfrac{1}{2}\sin^2\theta\rangle = -\omega^2 A_+^2\langle\sin 2\theta\rangle - \tfrac{1}{2}\omega^2 A_+^2\langle\sin^2\theta\rangle \\ &= 0 - \tfrac{1}{2}\langle\dot{h}_+\dot{h}_+\rangle \end{aligned} \tag{5.58}$$

where $\theta = \omega t - \omega z$. In a similar way, you can show that $R_{zz} = R_{tt} = -R_{tz} = -R_{zt}$, and all other $R_{\mu\nu} = 0$.

5.4.1 Exercise: R_{tz}

The Diagonal Metric Worksheet's expansion for R_{tz} is

$$R_{tz} = -\tfrac{1}{2B}B_{03} - \tfrac{1}{2C}C_{03} + \tfrac{1}{4B^2}B_0B_3 - \tfrac{1}{4C^2}C_0C_3 + \tfrac{1}{4AB}A_3B_0 + \tfrac{1}{4AC}A_3C_0 + \tfrac{1}{4DB}D_0B_3 + \tfrac{1}{4DC}D_0C_3 \tag{5.59}$$

Use this to show that $R_{tz} = -R_{tt}$ through second order in h_+.

This means that

$$R = g^{\mu\nu}R_{\mu\nu} = -(1 - h^{tt})R_{tt} + (1 - h^{zz})R_{zz} = -(1 + 0)R_{tt} + (1 + 0)R_{tt} = 0 \tag{5.60}$$

in transverse-traceless gauge. So the effective energy density of an uprightly polarized gravitational wave is

$$T_{tt}^{GW} = -\frac{\langle G_{tt}^{(2)}\rangle}{8\pi G} = -\frac{\langle R_{tt}^{(2)}\rangle}{8\pi G} = +\frac{\langle\dot{h}_+\dot{h}_+\rangle}{16\pi G} \tag{5.61}$$

The energy contributed by a diagonally polarized wave is trickier to calculate (we can't use the Diagonal Metric Worksheet), But the result cannot be any different than the above, because we can convert an uprightly polarized wave to a diagonally polarized wave simply by rotating coordinates by 45° around the z axis. Therefore, the formula for the total energy density of an arbitrary gravitational wave moving in the z direction must be

$$T_{tt}^{GW} = \frac{1}{16\pi G}\langle\dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times\rangle \tag{5.62}$$

We can write this more generally in the form

$$T_{tt}^{GW} = \frac{1}{32\pi G}\langle\dot{h}_{jk}^{TT}\dot{h}_{TT}^{jk}\rangle \tag{5.63}$$

(Since this sums over A_{xx}^2 *and* A_{yy}^2 and also A_{xy}^2 *and* A_{yx}^2, we get a factor of 2 in the inside the brackets that must be canceled by another factor of 2 in the denominator.) The advantage of this final expression is that the quantity inside the brackets is a scalar with respect to rotations in 3-space, and therefore will not depend on what direction the waves are moving, as long as we are in transverse-traceless gauge for whatever direction that is.

Finally, the gravitational wave energy flux (energy per unit time per unit area) in the z direction is

$$T^{tz}_{GW} = -T^{GW}_{tz} = +\frac{\langle R^{(2)}_{tz}\rangle}{8\pi G} = -\frac{\langle R^{(2)}_{tt}\rangle}{8\pi G} = \frac{1}{32\pi G}\langle \dot{h}^{TT}_{jk}\dot{h}^{jk}_{TT}\rangle = T^{GW}_{tt} \quad (!) \tag{5.64}$$

5.4.2 Exercise: Why is Energy Flux = Energy Density?

Explain physically why a gravitational wave energy flux *should* have the same magnitude as the gravitational wave energy density in units where $c = 1$. (*Hint:* Consider a surface of area A perpendicular to the wave. What volume of gravitational wave energy will go through that area in a time interval Δt?)

5.5 Source Luminosities.

We now know how to calculate the gravitational waves moving in the $+z$ direction generated by a given source, and we also know how to calculate the energy flux involved in such waves. This should give us what we need to know to calculate the total luminosity of a gravitational wave source, right?

The remaining problem is that gravitational waves from a given source will be radiated in all directions, and right now we only know how to calculate the transverse-traceless components for a wave moving in the $+z$ direction. We need to find how to find the components transverse-traceless components for a wave moving in an arbitrary direction $\vec{n}$ in a fixed coordinate system.

Conceptually, this is actually not difficult. We saw that for waves moving in the $+z$ direction, we are able to simply project the the solution $H^{\mu\nu}$ onto a spatial plane perpendicular to the direction of motion, and then subtract an equal portion of the matrix's remaining trace from every nonzero diagonal element (to make the matrix traceless). It is actually quite easy to do this for a direction parallel to any coordinate axis.

5.5.1 Exercise. TT Components for the $+x$ Direction.

Consider an arbitrary amplitude matrix of the form $A^{\mu\nu}$ for a gravitational wave moving in the $+x$ direction. Using the operations described above, find the $A^{\mu\nu}_{TT}$ components for that wave.

The trick to doing this for an arbitrary direction $\vec{n}$ is to express these operations in terms of 3-tensor operators that will therefore give the correct components in any coordinate system rotated with respect to our base system. It turns out that the tensor operator

$$P^i_j \equiv \delta^i_j - n^i n_j \tag{5.65}$$

(which, please note, is constructed entirely of 3-tensors) will project a vector on the plane perpendicular to the unit vector $\vec{n}$. When $\vec{n}$ is a unit vector in the z direction, this projection tensor has the value

$$P^i_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.66}$$

which should certainly do the job.

So since a second-rank tensor ought to behave like the tensor product of two vectors, the projection of a 3-tensor (say I^{ij}) should be simply $P^i_m P^j_n I^{mn}$. Finally, the trace of the projected matrix should be

$$\begin{aligned} I &= \eta_{lk}(P^l_m P^k_n I^{mn}) = P_{mk}P^k_n I^{mn} = (\eta_{mk} - n_m n_k)(\delta^k_n - n^k n_m)I^{mn} \\ &= (\eta_{mn} - n_m n_n - n_m n_n + n_m n_k n^k n_n)I^{mn} = (\eta_{mn} - n_m n_n)I^{mn} = P_{mn}I^{mn} \end{aligned} \tag{5.67}$$

since $n^k n_k = \vec{n}\cdot\vec{n} = 1$. (Remember that in this effectively flat spacetime, raising and lowering spatial indices does nothing: $n_i = \eta_{ij}n^j = (+1)n^i$.) Multiplying half of this times the projection matrix itself should be a matrix having nonzero diagonal elements perpendicular to $\vec{n}$ that are both half the trace of the projected tensor. If we then subtract this from the transformed tensor, we should have the transverse-traceless components. So the complete transformation operator should be

$$I_{TT}^{jk} = (P^i_m P^j_n - \tfrac{1}{2}P^{ij}P_{mn})I^{mn} \tag{5.68}$$

The energy flux of a gravitational wave in an arbitrary direction is $\langle \dot{h}^{jk}_{TT}\dot{h}^{TT}_{jk}\rangle/32\pi G$, and we also know that $h^{ij}_{TT} = (2GM/R)\ddot{I\!\!\!-}^{ij}_{TT}$. Substituting the latter into the former yields

$$\text{flux} = \frac{G}{8\pi R^2}\langle \dddot{I\!\!\!-}^{ij}_{TT}\dddot{I\!\!\!-}^{TT}_{ij}\rangle \tag{5.69}$$

for the flux in a particular direction (note the triple time-derivative of the reduced quadrupole moment tensor!). We can now substitute in equation 5.68 to show (after a *lot* of work) that

$$\text{flux in } \vec{n}\text{-direction} = \frac{G}{16\pi R^2}\langle 2\dddot{I\!\!\!-}^{ij}\dddot{I\!\!\!-}_{ij} - 4n^i n^j \dddot{I\!\!\!-}_i{}^m\dddot{I\!\!\!-}_{mj} + n^i n^j n^m n^n \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}_{mn}\rangle \tag{5.70}$$

The advantage of this equation is that we can calculate the reduced quadrupole moment tensor in whatever coordinate system we want, and calculate the flux of gravitational waves in any direction that we want.

We can find the total energy radiated by the source by computing the energy radiated per unit time through a differential area element $dA = R^2 \sin\theta\, d\theta\, d\phi$ on the surface of a sphere of radius R much larger than the source in a direction specified by the unit 3-vector $\vec{n}$ whose components are $n^x = \sin\theta\cos\phi, n^y = \sin\theta\sin\phi$, and $n^z = \cos\theta$, and then integrating that result over the entire sphere. The energy radiated through the area element is simply the flux times dA, so the integral will yield the rate at which the source is radiating. This in turn should be equal to the rate at which the source's total energy E is decreasing due to the energy carried away by gravitational radiation. So in summary, we have

$$\begin{aligned} -\frac{dE}{dt} &= \frac{G}{16\pi R^2}\int_0^\pi \left(\int_0^{2\pi} \langle \dddot{I\!\!\!-}^{ij}_{TT}\dddot{I\!\!\!-}^{TT}_{ij}\rangle\, d\phi\right) R^2 \sin\theta\, d\theta \\ &= \frac{G}{16\pi}\int_0^\pi \left(\int_0^{2\pi} \langle 2\dddot{I\!\!\!-}^{ij}\dddot{I\!\!\!-}_{ij} - 4n^i n^j \dddot{I\!\!\!-}_i{}^m\dddot{I\!\!\!-}_{mj} + n^i n^j n^m n^n \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}_{mn}\rangle\, d\phi\right)\sin\theta\, d\theta \end{aligned} \tag{5.71}$$

Now, in the last expression, the components of the reduced quadrupole moment tensor $\dddot{I\!\!\!-}^{ij}$ depend only on the orientation and behavior of the source in our coordinate system: they have nothing to to with the direction $\vec{n}$ that we are integrating over. Therefore, we can split this integral up into three parts and pull the terms involving the quadrupole moment tensor out in front of the integral:

$$\begin{aligned} -\frac{dE}{dt} = \frac{2G}{16\pi}\langle \dddot{I\!\!\!-}^{ij}\dddot{I\!\!\!-}_{ij}\rangle \int_0^\pi \left(\int_0^{2\pi} d\phi\right)\sin\theta\, d\theta - \frac{4G}{16\pi}\langle \dddot{I\!\!\!-}_i{}^m\dddot{I\!\!\!-}_{mj}\rangle \int_0^\pi \left(\int_0^{2\pi} n^i n^j\, d\phi\right)\sin\theta\, d\theta \\ + \frac{G}{16\pi}\langle \dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}_{mn}\rangle \int_0^\pi \left(\int_0^{2\pi} n^i n^j n^m n^n\, d\phi\right)\sin\theta\, d\theta \end{aligned} \tag{5.72}$$

Each remaining integral is simply a number whose value may depend on the choice of indices, but which is relatively easy to evaluate. When all the dust settles, the result is simply

$$-\frac{dE}{dt} = \frac{G}{5}\langle \dddot{I\!\!\!-}^{ij}\dddot{I\!\!\!-}_{ij}\rangle \tag{5.73}$$

This is an important and useful result.

5.6 Gravitational Waves from Binary Stars.

The most *common* cosmic sources of gravitational waves are binary systems. As a first approximation, let's treat the binary system as a pair of point masses m_1 and $m_2 \geq m_1$ separated by a fixed distance D. Let's set up a coordinate system so that the plane of the system's rotation is the xy plane. The orbital radii of the two masses are then

$$r_1 = \left(\frac{m_2}{m_1+m_2}\right)D \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1+m_2}\right)D \tag{5.74}$$

respectively. (Note that $r_1/r_2 = m_2/m_1$, which is appropriate if these are distances from the center of mass at the origin, and also that $r_1 + r_2 = D$.) Let's also define $t = 0$ to be the instant when mass m_1 crosses the $+x$ axis. Then the coordinates x_1, y_1 and x_2, y_2 at an arbitrary time t are

$$x_1 = r_1 \cos\omega t = \frac{m_2 D}{m_1 + m_2}\cos\omega t \qquad \text{and} \qquad y_1 = r_1 \sin\omega t = \frac{m_2 D}{m_1 + m_2}\sin\omega t \tag{5.75a}$$

$$x_2 = -r_2 \cos\omega t = -\frac{m_1 D}{m_1 + m_2}\cos\omega t \qquad \text{and} \qquad y_2 = -r_2 \sin\omega t = -\frac{m_1 D}{m_1 + m_2}\sin\omega t \tag{5.75b}$$

where ω is the orbital angular frequency. Components of the reduced quadrupole moment tensor are therefore

$$\not{I}^{xx} = \int_{\text{src}} \rho(x^2 - \tfrac{1}{3}\eta^{xx}r^2)\,dV = m_1(x_1^2 - \tfrac{1}{3}r_1^2) + m_2(x_2^2 - \tfrac{1}{3}r_2^2) = (m_1 r_1^2 + m_2 r_2^2)(\cos^2\omega t - \tfrac{1}{3}) \tag{5.76a}$$

$$\not{I}^{xy} = \int_{\text{src}} \rho(xy - \tfrac{1}{3}\eta^{xy}r^2)\,dV = m_1(x_1 y_1 - 0) + m_2(x_2 y_2 - 0) = (m_1 r_1^2 + m_2 r_2^2)\cos\omega t\sin\omega t \tag{5.76b}$$

Similarly, $\not{I}^{yy} = (m_1 r_1^2 + m_2 r_2^2)(\sin^2\omega t - \frac{1}{3})$, $\not{I}^{zz} = \frac{1}{3}(m_1 r_1^2 + m_2 r_2^2)$ and all other $\not{I}^{ij} = 0$. We can simplify this by using the double-angle trigonometric identities $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta, \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, as well as the definitions

$$\eta \equiv \frac{m_1 m_2}{(m_1 + m_2)^2} \quad \text{and} \quad M \equiv m_1 + m_2 \tag{5.77}$$

Note also that

$$\begin{aligned} m_1 r_1^2 + m_2 r_2^2 &= m_1\left(\frac{m_2 D}{m_1 + m_2}\right)^2 + m_2\left(\frac{m_1 D}{m_1 + m_2}\right)^2 \\ &= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2}D^2 = \frac{m_1 m_2 D^2 (m_1 + m_2)}{(m_1 + m_2)^2} = \eta M D^2 \end{aligned} \tag{5.78}$$

So our quadrupole moment tensor becomes

$$\not{I}^{ij} = \tfrac{1}{2}M\eta D^2 \begin{bmatrix} \frac{1}{3} + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \frac{1}{3} - \cos 2\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \tag{5.79}$$

(Note that $\frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3}$.) The double-time derivative of this is

$$\ddot{\not{I}}^{ij} = -2M\eta D^2\omega^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.80}$$

This matrix already happens to be in transverse-traceless gauge for radiation in the $+z$ direction, so for an observer in the $+z$ direction and a distance R from the system's center of mass, the metric perturbation is

$$h_{TT}^{ij} = H_{TT}^{ij} = \frac{2G}{R}\ddot{\not{I}}_{TT}^{ij} = -\frac{4GM\eta D^2\omega^2}{R}\begin{bmatrix} \cos 2\omega(t - R) & \sin 2\omega(t - R) & 0 \\ \sin 2\omega(t - R) & -\cos 2\omega(t - R) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.81}$$

Note that $A_+ = A_\times = -4GM\eta D^2\omega^2/R$, so this wave has equal amounts of plus and cross polarization. Also note the plus and cross polarizations are $90°$ out of phase, implying that wave is *circularly* polarized (the ring-distortion ellipse will rotate counterclockwise instead of oscillating in and out). Finally note that the wave has a frequency that is *twice* the orbital frequency of the system.

To find the gravitational waves radiated in another direction, we can use the general projection operator

$$\ddot{\not{I}}_{TT}^{jk} = (P_m^i P_n^j - \tfrac{1}{2}P^{ij}P_{mn})\ddot{\not{I}}^{jk} \quad \text{with} \quad P_j^i = \delta_j^i - n^i n_j \tag{5.82}$$

or in simple cases, we can just carry out the operations by eye. For example, the expression for the case of waves moving in the $+x$ direction is simply

$$h_{TT}^{ij} = \frac{2GM\eta D^2\omega^2}{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos 2\omega(t - R) & 0 \\ 0 & 0 & -\cos 2\omega(t - R) \end{bmatrix} \tag{5.83}$$

This wave is purely plus-polarized and A_+ has half the magnitude it had for waves moving in the $+z$ direction. This result is important, because many of the binaries we know about are eclipsing binaries, so we will be viewing such binaries from this angle.

5.6.1 Exercise: TT Components by Eye

Arrive at equation 5.83 by applying the following steps: (1) Project $\ddot{I}^{jk}$ given in equation 5.80 onto the plane perpendicular to the x direction, (2) calculate the trace of the remaining components, (3) subtract half the trace from each diagonal element not affected by the projection, and (4) evaluate at the retarded time.

The other point that you ought to take away from this example is that both the amplitude of the wave and the relative magnitudes of its polarizations will depend on your viewing angle.

The total power radiated in all directions is given by

$$-\frac{dE}{dt} = \frac{G}{5}\langle \dddot{I}^{ij}\dddot{I}_{ij}\rangle = \frac{32(GM)^2\eta^2 D^4\omega^6}{5G} \tag{5.84}$$

Note the astonishing 6th-power dependence of this energy loss on the rotational frequency!

5.6.2 Exercise: The Energy Loss Formula.

You can actually derive this result pretty easily from equation 5.80. Explain how. In particular, where does the factor of 32 come from?

Finally, let's see what effect this has on the system itself. Assume that the stars move slowly enough that their velocities are non-relativistic, and that they are far enough apart that Newtonian gravitational theory is adequate to predict their motion. Newton's second law applied to the star with mass m_1 tells us that

$$\frac{Gm_1m_2}{D^2} = \frac{m_1v_1^2}{r_1} \quad\Rightarrow\quad \frac{Gm_2}{D^2} = r_1\left(\frac{v_1}{r_1}\right)^2 = \frac{m_2D}{M}\omega^2 \quad\Rightarrow\quad D^3 = \frac{GM}{\omega^2} \tag{5.85}$$

We can use this to eliminate the usually unmeasurable quantity D in favor of the much more easily measured orbital frequency ω. Substituting this back into the luminosity equation gives

$$-\frac{dE}{dt} = \frac{32(GM)^2\eta^2\,\omega^6}{5G}\left(\frac{GM}{\omega^2}\right)^{4/3} = \frac{32\eta^2}{5G}(GM\omega)^{10/3} \tag{5.86}$$

This shows that the rate of energy loss increases dramatically as the system's total mass increases and/or its orbital frequency increases.

This energy must come at the expense of the system's orbital energy which one can show is equal to

$$E = -\frac{Gm_1m_2}{2D} = -\frac{G(\eta M^2)\omega^{2/3}}{2(GM)^{1/3}} = -\tfrac{1}{2}M(GM\omega)^{2/3}\eta \tag{5.87}$$

We see that stars in the binary pair will maintain neither a fixed separation D nor a constant angular frequency ω, as assumed in the derivation: rather ω will increase with time (and D will decrease) as the binary's orbital energy is radiated away. This means that the calculations we have made are not quite right: for example, our calculation for $\ddot{I}^{ij}$ is not exact because we are ignoring the time dependence of both D and ω. However, as long as the energy leaks away only very slowly, we are justified in ignoring these time derivatives.

One way to quantify how "slowly" the energy is radiated is to calculate the time rate of change of the orbit's period $T = 2\pi/\omega$. Note that since

$$E = -\tfrac{1}{2}M(GM\omega)^{2/3}\eta \quad\Rightarrow\quad dE = -\tfrac{1}{3}M(GM)^{2/3}\omega^{-1/3}\eta\,d\omega \quad\Rightarrow\quad \frac{d\omega}{dE} = -\frac{3\,\omega^{1/3}}{M(GM)^{2/3}\eta} \tag{5.88}$$

Therefore, the orbital period's time rate of change is

$$\frac{dT}{dt} = \frac{dT}{d\omega}\frac{d\omega}{dE}\frac{dE}{dt} = \left(-\frac{2\pi}{\omega^2}\right)\left(-\frac{3\,\omega^{1/3}}{M(GM)^{2/3}\eta}\right)\left(-\frac{32\eta^2}{5G}(GM\omega)^{10/3}\right) = \frac{192\pi\eta}{5}(GM\omega)^{5/3} \tag{5.89}$$

Note that in general, unless the objects are actually coalescing, the orbit's period T in meters of light travel time for a typical binary pair will be very large compared to GM for the pair, so $GM\omega \propto GM/T$ will be very small. This justifies our approximation that D and ω are approximately constant.

A different way of expressing how the energy loss affects the orbit is to directly calculate the rate at which the orbital *frequency* ω changes:

$$\frac{d\omega}{dt} = \frac{d\omega}{dE}\frac{dE}{dt} = \left(-\frac{3\,\omega^{1/3}}{M(GM)^{2/3}\eta}\right)\left(-\frac{32\eta^2}{5G}(GM\omega)^{10/3}\right) = \frac{96\eta}{5}(GM)^{5/3}\omega^{11/3} = \frac{96}{5}(G\mathcal{M})^{5/3}\omega^{11/3}$$

$$\text{where } \mathcal{M} \equiv \eta^{3/5}M = \left(\frac{m_1m_2}{[m_1+m_2]^2}\right)^{3/5}[m_1+m_2] = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}} \tag{5.90}$$

We call the quantity $\mathcal{M}$ the "chirp mass" of the binary, in that at any given orbital frequency, the rate at which the frequency increases is determined completely by $\mathcal{M}$.

As an example, consider the binary pair known as HM Cancri (RX JO806.3+1527). This system consists of two white-dwarf stars with masses of 0.55 $M_\odot$ and 0.27 $M_\odot$ (M = 1200 m, $\eta = 0.149/0.82^2 = 0.22$) orbiting with a period of 321.53 s $= 9.6\times10^{10}$ m. The distance to this system is not well known but is probably close to 16,000 ly $= 1.5\times10^{20}$ m. For this system, $GM\omega = 7.9\times10^{-8}$, and from that, one can show that for this fairly face-on system (inclination angle 38°),

$$A_+ \approx \frac{4GM\eta D^2\omega^2}{R} = \frac{4GM\eta}{R}(GM\omega)^{2/3} = 1.3\times10^{-22} \tag{5.91a}$$

$$-\frac{dE}{dt} = \frac{32\eta^2}{5G}(GM\omega)^{10/3} = 2.4\times10^{28}\,\mathrm{W} \tag{5.91b}$$

$$\frac{dT}{dt} = \frac{192\pi\eta}{5}(GM\omega)^{5/3} = -3.9\times10^{-11} \tag{5.91c}$$

The gravitational wave power radiated by the system is fairly large: about 60 times the rate that the sun radiates EM energy. However, this is such a small fraction of the system's total energy that the rate of change of the period is so small as to be difficult to detect (1.2 ms per year). To detect waves from this system, a gravitational wave detector would need to be able to measure fractional changes in the distance between floating masses of at least 10^{-22} at a frequency of 2/321.53 s = 6.2 mHz. The LIGO detector has the appropriate sensitivity only in a frequency range of hundreds of Hz, not in this low frequency range. But this system's frequency is at the sweet spot for LISA, and would have a predicted signal-to-noise ratio of better than 200 for a four-year observation period. This is one of about 50 candidate "LISA verification binaries" detailed in a very recent paper. [2]

Before 2015, the strongest evidence available for the existence of gravitational waves was observations of the Hulse-Taylor binary system (PSR B1913+16), which includes a pulsar. Because a pulsar is an extraordinarily good clock, one is able to infer this system's orbital parameters to extraordinary accuracy from the Doppler shifts of the pulsar's signal. Though this system is complicated to analyze (because the orbit fairly elliptical). This system has been observed for more than 40 years, and energy loss from this system over this period is very clear, and the ratio of the observed loss to the accumulated loss predicted by general relativity is currently 0.9983 ± 0.0016.[3]

5.6.3 Exercise: Power Radiated by HM Cnc.

Verify the power calculation given above. The value of G in units where $c = 1$ is 7.426×10^{-28} m/kg, and after using this to get the radiated power in kg (of energy) per meter (of time), multiply the result by the appropriate power of c to convert to watts. What is that power of c?

As you can see, these calculations assume a completely Newtonian source, an approximation that gets worse and worse as a binary system approaches coalescence. In the HM Cancri system above, the speeds of the orbiting white dwarfs are about 400 km/s = 0.0013c, so it is possible that relativistic effects would be measurable. To be able to better predict the waveforms from weakly-relativistic binary systems, a research group associated with Luc Blanchet did calculations in the 1990s extending the work that we have rehearsed here by calculating correction terms to higher orders in v/c. This leads to a "Post-Newtonian" expansion of the gravitational wave from a binary pair in a circular orbit that looks something like [4]

$$h_+ = \frac{2GM\eta}{R}(GM\omega)^{2/3}(B_+^{(0)} + y^{1/2}B_+^{(1/2)} + y^1B_+^{(1)} + y^{3/2}B_+^{(3/2)} + y^2B_+^{(2)}) \tag{5.92}$$

where $y \equiv (GM\omega)^{2/3} \approx GM/D \approx v^2$, where

$$B_+^{(0)} = -(1+\cos^2 i)\cos 2\Psi, \quad B_+^{(1/2)} = -\frac{\sin i}{8}\frac{\delta m}{M}\left[(5+\cos^2 i)\cos\Psi + 9(1+\cos^2 i)\cos 3\Psi\right] \tag{5.93}$$

and so on, where i is the inclination angle between the orbit's normal and the line of sight, and Ψ is the orbital phase, whose leading term is $\omega(t - R)$ but also which has correction terms. A similar expansion yields the cross polarization. (In this formulation, the inclination-angle dependence arises from projecting the transverse- traceless components onto the plane perpendicular to the line of sight.) We see that this formula represents basically an expansion in a power series in the orbital speed v. The series above carries this expansion out to a factor of $(GM/D)^2 \approx v^4$, which people describe as being to "post-Newtonian order 2" or 2PN. The additional terms come mostly from higher-order corrections to the motions of the binary stars coming from the geodesic equations of motion.

I am not going to go over this approach in any depth, but I thought that you should know about it. Note that the additional terms add harmonics to the waveform (up to 6 times the orbital frequency in the 2PN terms) that convey additional information about the source.

My research over the past few decades has been in trying to model what LISA could learn about various sources considering the additional information conveyed in the harmonics and comparing that to what one would learn by focusing only on the fundamental wave. In certain cases, we have found that the harmonics can resolve degeneracies that would arise in the basic waveform, allowing one to determine source parameters that would otherwise be uncertain. To this end, I have developed a computer application which, given the parameters for a specific binary source, calculates the gravitational wave produced by that source, models its detection by LISA (observing over a year) and then estimates what uncertainties in the parameter values will result from noise in the LISA detector.

Currently, I am working with a student to add in the effects that the spin-spin and spin-orbit interactions have on the waveform and phase. This has involved rewriting the program almost completely, as many quantities that we could calculate analytically before now have to be calculated numerically. We are currently in the process of testing the program to ensure that it is giving the same results as the initial program under the same circumstances (a process that, in the way research usually goes, has uncovered some issues with the original program as well). We hope to be producing a publication soon about this research. I am also happy to talk in more depth about this particular research project with anyone who is interested.

We are finally at the end! I hope that you have enjoyed this brief overview of the path from the basic principles of GR through its practical application to gravitational waves. I will be around for the rest of the month and would be happy to answer any questions you might have about what I have presented and/or the homework problems. Thank you for your attention, and the honor of presenting to you here.

Homework Problems

5.1 Here is the other approach to arriving at transverse-traceless gauge. In this problem we will actually derive a coordinate transformation that will force an arbitrary metric perturbation satisfying the Einstein equation and the Lorenz condition into transverse-traceless form. Consider a coordinate transformation of the form $\xi^\mu = B^\mu \sin k_\sigma x^\sigma$, where B^μ are a set of four undetermined constants.

- **(a)** Show that this gauge transformation satisfies the condition $\Box^2 \xi^\mu = 0$ that we must obey if the transformation is to preserve the Lorenz gauge condition.
- **(b)** If our original metric perturbation is $H^{\mu\nu} = A^{\mu\nu} \cos k_\sigma x^\sigma$, show that the new transformed A-matrix satisfies $A'^{\mu\nu} = A^{\mu\nu} - k^\mu B^\nu - k^\nu B^\mu + \eta^{\mu\nu} k_\alpha B^\alpha$
- **(c)** To make the wave "transverse", we want to set all components of the wave that involve a t index to zero: $0 = A'^{t\nu} = A^{t\nu} - k^t B^\nu - k^\nu B^t + \eta^{t\nu} k_\alpha B^\alpha$. Show that the "traceless" condition requires that $0 = A'^\mu{}_\mu = A^\mu{}_\mu + 2k_\mu B^\mu$.
- **(d)** If we solve the latter equation for $k_\mu B^\mu$ and plug it into the other, it becomes $0 = A'^{t\nu} = A^{t\nu} - k^t B^\nu - k^\nu B^t - \frac{1}{2}\eta^{t\nu} A^\mu{}_\mu$. For the particular case of $k_t = -\omega, k_x = k_y = 0, k_z = \omega$, solve this set of four equations for four coefficients B^t, B^x, B^y, and B^z in terms of whatever the original values of $A^{\mu\nu}$ were that will make this equation true. (Partial answer: $B^z = \omega^{-1}[A^{tz} - \frac{1}{2}A^{tt} - \frac{1}{4}A^\mu{}_\mu]$).
- **(e)** Then argue that the Lorenz gauge condition applied to transformed matrix $A'^{\mu\nu}$ also requires that $A'^{z\nu} = 0$, ensuring that the transformed matrix is completely transverse.

(Note that this does not quite exclude the possibility that another transformation could erase the remaining nonzero elements. The argument that I presented based on the Riemann tensor does exclude that possibility: we cannot transform coordinate in a way that changes a curved spacetime into a flat spacetime.)

5.2 Consider a rigid rod that separates two small spheres of mass m so that they are a distance L apart along the X axis in a LOF observation frame. (I am using X to distinguish the LOF frame coordinate from the comoving TT-coordinate x, but assume that these axes point in the same direction.)Define the origin of that axis so that the centers of the spheres are at $X = \pm\frac{1}{2}L$. Suppose a plus-polarized gravitational wave with amplitude A_+ and angular frequency ω passes through this object. Determine the X component of the force that the rod must exert on each mass to hold it in place. (*Hint:* We know that the distances between freely-floating objects will change as the gravitational wave passes, so objects that are held a fixed distance apart are not following geodesics.)

5.3 Calculate the flux (in W/m^2) of a plus-polarized gravitational wave with an amplitude of $A_+ = 10^{-20}$ and a frequency of 100 Hz (the sort of gravitational wave that LIGO might detect). Are you surprised?

5.4 Consider two small but dense objects of mass m connected by a massless spring with zero relaxed length. The masses are oscillating back and forth along the x axis (barely missing each other at the origin) with amplitude D and angular frequency ω.

(a) Calculate the reduced quadrupole moment tensor $\rlap{\,/}I^{ij}$ for this system.

(b) Calculate the transverse-traceless gravitational wave components as observed by an observer a distance R away in the $+z$ direction. Is the wave plus-polarized, cross-polarized or a combination.

(c) Calculate the transverse-traceless gravitational wave components as observed by an observer a distance R away in the $+y$ direction.

(d) Calculate the transverse-traceless gravitational wave components as observed by an observer a distance R away in the $+x$ direction. Are you surprised by the result?

5.5 Estimate the rate at which the orbiting Earth radiates energy in the form of gravitational waves. Is it likely that the earth will spiral into the sun any time soon as a result of this energy loss?

5.6 PSR J0737-3039 is a recently discovered binary pulsar system consisting of two neutron stars, one with a mass of 1.337 $M_\odot$ and the other with a mass of 1.250 $M_\odot$, orbiting with a period of 2.4 h. Their orbit is only mildly eccentric, and we are viewing the orbit almost edge-on. This system is about 1800 ly away. Find **(a)** the value of $GM\omega$ for this system, **(b)** the gravitational wave amplitude A_+ at the Earth, **(c)** the total gravitational wave power this object emits in watts, and **(d)** the rate dT/dt at which its period is changing.

5.7 If you love algebra, show that equation 5.70 follows from equations 5.69 and 5.68 as well as the definition of the projection operator 5.65. (There is nothing horribly difficult about this, but one must be careful to keep track of all the terms and all the indices.)

5.8 Consider the integral

$$\int_0^\pi \left(\int_0^{2\pi} n^i n^j \, d\phi\right) \sin\theta \, d\theta \tag{5.94}$$

(a) Argue that the integral will be zero if $i \neq j$.

(b) Argue physically that the integral must have the same value if $i = j$, no matter what the value of i might be. Therefore, the result will be proportional to η^{ij}.

(c) Pick one integral to do to show that this quantity is in fact $\frac{4\pi}{3}\eta^{ij}$.

(d) Use a similar argument to show that

$$\int_0^\pi \left(\int_0^{2\pi} n^i n^j n^m n^n \, d\phi\right) \sin\theta \, d\theta = \frac{4\pi}{15}(\eta^{ij}\eta^{mn} + \eta^{jm}\eta^{in} + \eta^{im}\eta^{jn}) \tag{5.95}$$

(e) Use this to arrive at equation 5.73.

Notes

[1]Moore, A General Relativity Workbook, University Science Books, 2013, pp. 267-268.

[2]Kupfer et al. "LISA verification binaries with updated distances from *Gaia* Data Release 2", arXiv:1805.00482v2, June 15, 2018.

[3]Weisberg and Huang,"Relativistic Measurements from Timing the Binary Pulsar PSR B1913+16", *Astrophysical Journal* 829:55, 2016.

[4]Blanchet, Iyer, Will, and Wiseman"Gravitational waves from inspiraling compact binaries to second post-Newtonian order", *Classical and Quantum Gravity*, **13** (1996) pp. 575-584.