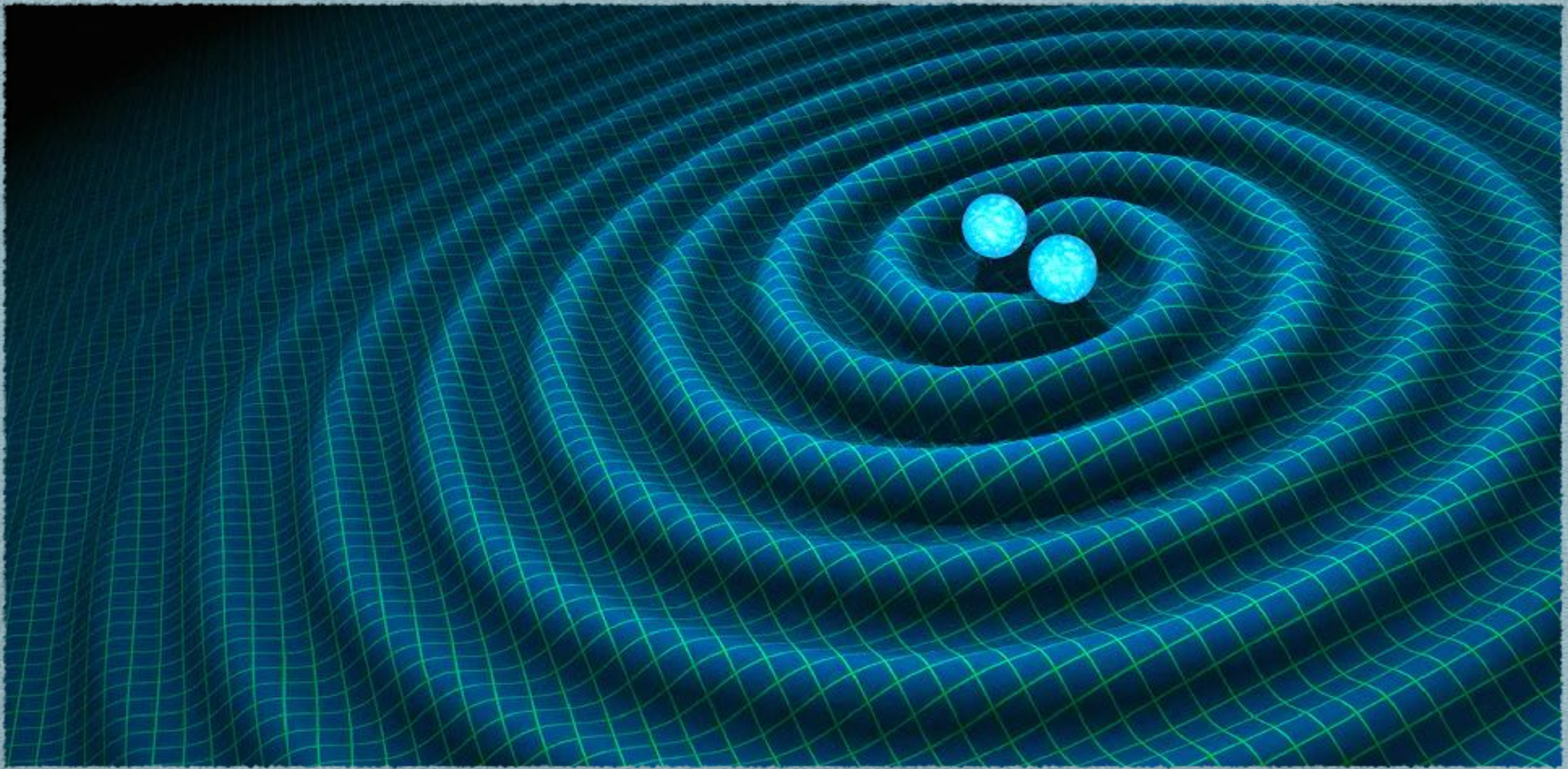


General Relativity and Gravitational Waves:

Session 5. Gravitational Waves

Thomas A. Moore — Les Houches — July 9, 2018

Overview of this session:

4.2 Transverse-Traceless Gauge

4.3 Generating Gravitational Waves

4.4 Gravitational Wave Energy

4.5 Source Luminosity

4.6 Gravitational Waves from Binary Stars

The Transverse-Traceless Gauge: Fundamentals

We start where we left off last time:

Weak-Field Approximation: $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$

Trace-reversed Perturbation: $H^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$

Einstein Equation: $\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu}$

Lorenz condition: $\partial_\mu H^{\mu\nu} = 0$

Remaining gauge freedom:

$$x'^{\,\alpha} = x^\alpha + \xi^\alpha \quad \text{where } |\xi^\alpha| \ll 1 \quad \text{and } \ \Box^2 \xi^\alpha = 0$$

Gauge transformation:

$$H'^{\,\mu\nu} = H^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\mu \xi^\nu + \eta^{\mu\nu} \partial_\alpha \xi^\alpha$$

The Transverse-Traceless Gauge: Fundamentals

In empty space, attempt a solution of the form:

$$H^{\mu\nu} = A^{\mu\nu} \cos k_\sigma x^\sigma = A^{\mu\nu} \cos(\vec{k} \cdot \vec{r} - \omega t)$$

Einstein Equation: $\Box^2 H^{\mu\nu} = 0 \quad \Rightarrow \quad k^\alpha k_\alpha = 0$

Lorenz condition: $\partial_\mu H^{\mu\nu} = 0 \quad \Rightarrow \quad k_\mu A^{\mu\nu} = 0$

Symmetry: $\Rightarrow \quad A^{\mu\nu} = A^{\nu\mu}$

The Einstein equation implies that

$$0 = k^\alpha k_\alpha = \eta^{\alpha\beta} k_\alpha k_\beta = \eta^{tt}(-\omega)^2 + \eta^{xx}(k_x)^2 + \eta^{yy}(k_y)^2 + \eta^{zz}(k_z)^2$$

$$\Rightarrow \quad 0 = -\omega^2 + k^2 \quad \Rightarrow \quad \omega = k \quad \Rightarrow \quad v = \frac{\omega}{k} = 1$$

The Transverse-Traceless Gauge: How can we tell what is real?

Real waves curve spacetime, and are thus revealed by the Riemann tensor

$$
\begin{array}{cccccccc}
 & \mu\nu \rightarrow & tx & ty & tz & xy & xz & yz \\
\alpha\beta\downarrow & tx & R_{txtx} & R_{txty} & R_{txtz} & R_{txxy} & R_{txxz} & R_{txyz} \\
 & ty & & R_{tyty} & R_{tytz} & R_{tyxy} & R_{tyxz} & R_{tyyz} \\
 & tz & & & R_{tztz} & R_{tzxy} & R_{tzxz} & R_{tzyz} \\
 & xy & & & & R_{xyxy} & R_{xyxz} & R_{xyyz} \\
 & xz & & & & & R_{xzxz} & R_{xzyz} \\
 & yz & & & & & & R_{yzyz}
\end{array}
$$

Assume $k_t = -\omega$, $k_x = k_y = 0$, $k_z = \omega$. Lorenz condition then implies

$$0 = k_\mu A^{\mu\nu} = -\omega A^{t\nu} + \omega A^{z\nu} \quad \Rightarrow \quad
\begin{aligned}
A^{tt} &= A^{zt} \ (= A^{tz}) \\
A^{tx} &= A^{zx} \ (= A^{xt} = A^{xz}) \\
A^{ty} &= A^{zy} \ (= A^{yt} = A^{yz}) \\
A^{tz} &= A^{zz} \ (= A^{zt} = A^{tt} \text{ from above})
\end{aligned}$$

The Transverse-Traceless Gauge: How can we tell what is real?

Also note that

$$\partial_\beta \partial_\mu h_{\alpha\nu} = \partial_\beta \partial_\mu (H_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu} H) = k_\beta k_\mu (A_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu} A) \sin k_\sigma x^\sigma$$

with $\quad A \equiv \eta_{\mu\nu} A^{\mu\nu} = -A^{tt} + A^{xx} + A^{yy} + A^{zz} = A^{xx} + A^{yy}$

Finally, note that

$$A_{\alpha\nu} = \eta_{\alpha\beta}\eta_{\nu\mu} A^{\beta\mu} = \begin{cases} -A^{\alpha\nu} \text{ if } \textit{either } \alpha = t \text{ or } \nu = t \text{ but not both} \\ +A^{\alpha\nu} \text{ otherwise} \end{cases}$$

Riemann tensor components are:

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \tfrac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}) \\ &= -\tfrac{1}{2}(k_\beta k_\mu[A_{\alpha\nu} - \tfrac{1}{2}\eta_{\alpha\nu}A] + k_\alpha k_\nu[A_{\beta\mu} - \tfrac{1}{2}\eta_{\beta\mu}A] \\ &\qquad - k_\alpha k_\mu[A_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}A] - k_\beta k_\nu[A_{\alpha\mu} - \tfrac{1}{2}\eta_{\alpha\mu}A]) \sin k_\sigma x^\sigma \end{aligned}$$

The Transverse-Traceless Gauge: How can we tell what is real?

Some specific Riemann tensor components:

$$\begin{aligned}
R_{txtx} &= -\tfrac{1}{2}\big(k_x k_t[A_{xt} - \tfrac{1}{2}\eta_{xt}A] + k_t k_x[A_{tx} - \tfrac{1}{2}\eta_{tx}A] \\
&\qquad - k_t k_t[A_{xx} - \tfrac{1}{2}\eta_{xx}A] - k_x k_x[A_{tt} - \tfrac{1}{2}\eta_{tt}A]\big)\sin k_\sigma x^\sigma \\
&= -\tfrac{1}{2}\big(0 + 0 - \omega^2[A_{xx} - \tfrac{1}{2}(A_{xx} + A_{yy})] - 0\big)\sin k_\sigma x^\sigma \\
&= +\tfrac{1}{4}\omega^2(A_{xx} - A_{yy})\sin k_\sigma x^\sigma
\end{aligned}$$

$$\begin{aligned}
R_{txtz} &= -\tfrac{1}{2}\big(k_x k_t[A_{tz} - \tfrac{1}{2}\eta_{tz}A] + k_t k_z[A_{xt} - \tfrac{1}{2}\eta_{xt}A] \\
&\qquad - k_t k_t[A_{xz} - \tfrac{1}{2}\eta_{xz}A] - k_x k_z[A_{tt} - \tfrac{1}{2}\eta_{tt}A]\big)\sin k_\sigma x^\sigma \\
&= -\tfrac{1}{2}\big(0 - \omega^2 A_{xt} - \omega^2 A_{xz} - 0\big)\sin k_\sigma x^\sigma \\
&= +\tfrac{1}{2}\omega^2(A_{xt} + A_{xz})\sin k_\sigma x^\sigma = 0
\end{aligned}$$

The Transverse-Traceless Gauge: How can we tell what is real?

The entire list of Riemann components

$\alpha\beta \downarrow$ \ $\mu\nu \rightarrow$	tx	ty	tz	xy	xz	yz
tx	$R_{txtx} = a$	$R_{txty} = b$	$R_{txtz} = 0$	$R_{txxy} = 0$	$R_{txxz} = a$	$R_{txyz} = b$
ty		$R_{tyty} = -a$	$R_{tytz} = 0$	$R_{tyxy} = 0$	$R_{tyxz} = b$	$R_{tyyz} = a$
tz			$R_{tztz} = 0$	$R_{tzxy} = 0$	$R_{tzxz} = 0$	$R_{tzyz} = 0$
xy				$R_{xyxy} = 0$	$R_{xyxz} = 0$	$R_{xyyz} = 0$
xz					$R_{xzxz} = a$	$R_{xzyz} = b$
yz						$R_{yzyz} = -a$

where $a \equiv \frac{1}{4}\omega^2(A_{xx} - A_{yy}) \sin k_\sigma x^\sigma$ and $b \equiv \frac{1}{2}\omega^2 A_{xy} \sin k_\sigma x^\sigma$.

(Note that $R_{txyz} + R_{tzxy} + R_{tyzx} = R_{txyz} + R_{tzxy} - R_{tyxz} = 0$.)

The Transverse-Traceless Gauge: How can we tell what is real?

The only values that matter: $A_{xx} - A_{yy}$ and $A_{yx} = A_{xy}$

We therefore ought to be able to do a coordinate transformation to erase all $A^{t\mu}$ and $A^{z\mu}$ and also make the matrix traceless:

$$A^{xx}_{\text{new}} = A^{xx} - \tfrac{1}{2}A = A^{xx} - \tfrac{1}{2}(A^{xx} + A^{yy}) = \tfrac{1}{2}(A^{xx} - A^{yy})$$
$$A^{yy}_{\text{new}} = A^{yy} - \tfrac{1}{2}A = A^{yy} - \tfrac{1}{2}(A^{xx} + A^{yy}) = -\tfrac{1}{2}(A^{xx} - A^{yy})$$

We call the gauge where waves in the +*z* direction have the form

$$H^{\mu\nu}_{TT} = \left(A_+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + A_\times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \cos k_\sigma x^\sigma$$

transverse-traceless gauge. Note that $h^{\mu\nu}_{TT} = H^{\mu\nu}_{TT} - \frac{1}{2}\eta^{\mu\nu} H_{TT} = H^{\mu\nu}_{TT}$

The Transverse-Traceless Gauge: Physical effects of the wave

The geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu}u^\mu u^\nu = -\Gamma^\alpha_{tt}u^t u^t = -\tfrac{1}{2}\eta^{\alpha\beta}(\partial_t h^{TT}_{t\beta} + \partial_t h^{TT}_{\beta t} - \partial_\beta h^{TT}_{tt})u^t u^t = 0\,!$$

Uh oh. Do the waves really have no physical effect?

The Transverse-Traceless Gauge: Physical effects of the wave

We have to check the metric! Consider a plus-polarized wave moving through a ring of floating particles such that $R^2 = \Delta x^2 + \Delta y^2$:

$$\begin{aligned}
\Delta s^2 &= (\eta_{xx} + h_{xx}^{TT})\Delta x^2 + (\eta_{yy} + h_{yy}^{TT})\Delta x^2 \\
&= (1 + A_+)R^2 \cos^2\theta \cos^2\omega t + (1 - A_+)R^2 \sin^2\theta \cos^2\omega t \\
&= R^2[1 + A_+(\cos^2\theta - \sin^2\theta)]\cos^2\omega t = R^2(1 + A_+\cos 2\theta)\cos^2\omega t
\end{aligned}$$

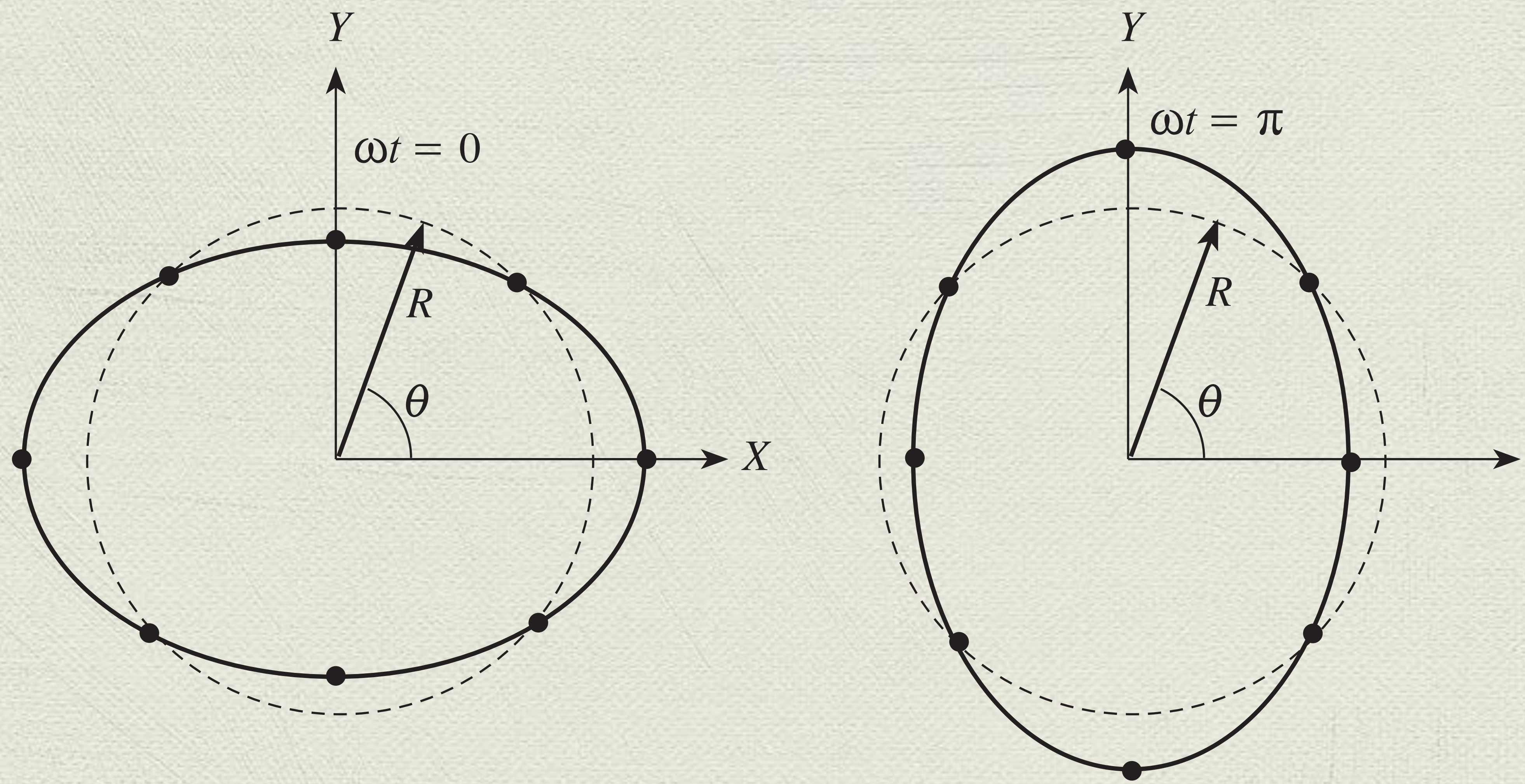
$$\Rightarrow \quad \Delta s = R(1 + A_+\cos 2\theta)^{1/2}\cos\omega t \approx R(1 + \tfrac{1}{2}A_+\cos 2\theta)\cos\omega t$$

Similarly, for a cross-polarized wave:

$$\Delta s \approx R(1 + \tfrac{1}{2}A_\times \sin 2\theta)\cos\omega t$$

The Transverse-Traceless Gauge: Physical effects of the wave

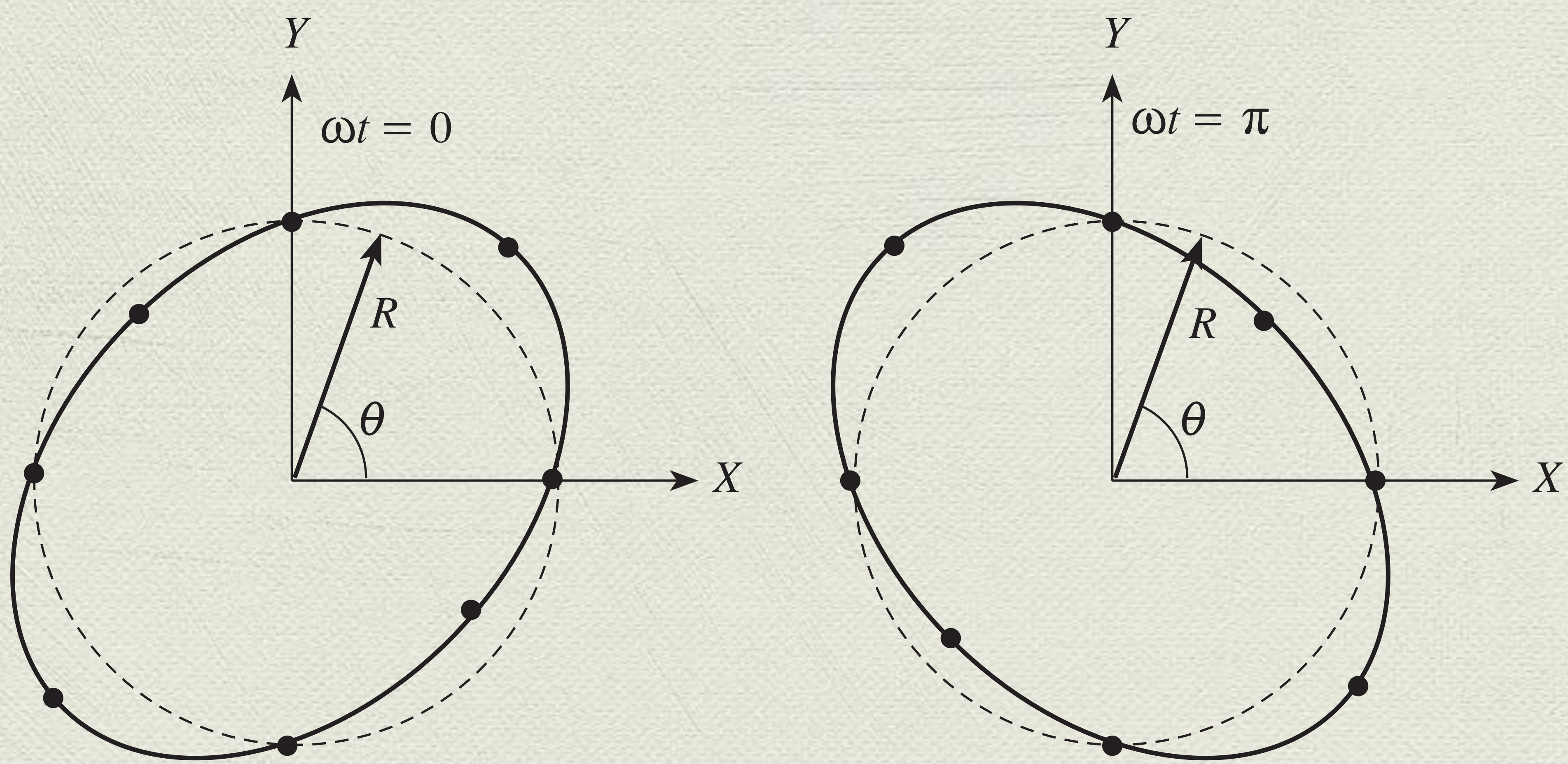
Upright (plus) polarization:



$$\Delta s \approx R(1 + \tfrac{1}{2}A_+ \cos 2\theta)\cos\omega t$$

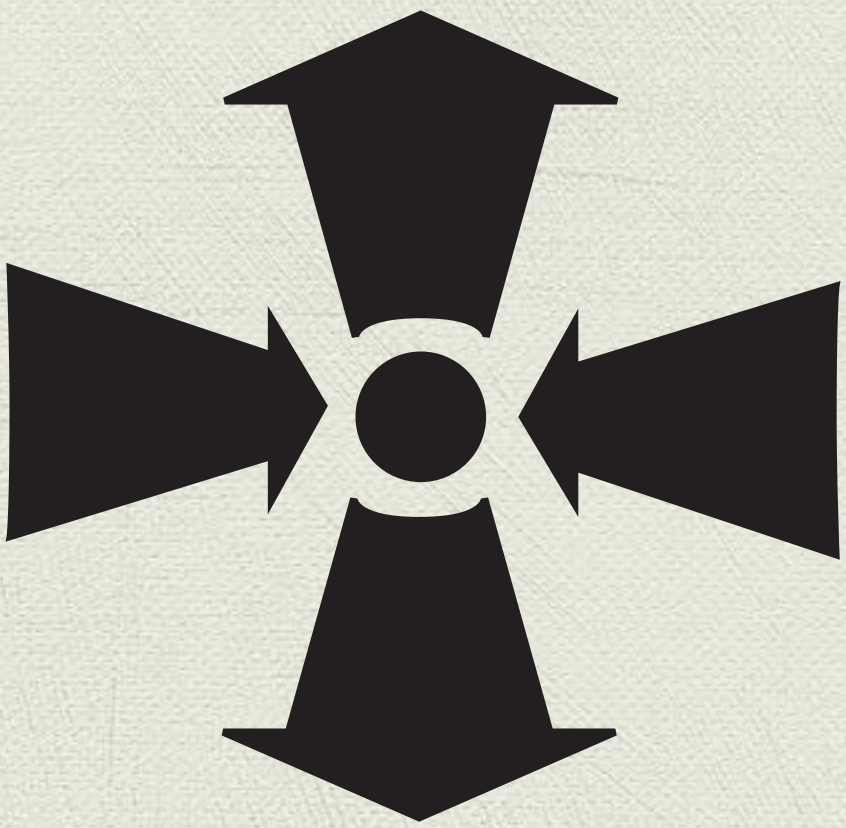
The Transverse-Traceless Gauge: Physical effects of the wave

Diagonal (cross) polarization:



$$\Delta s \approx R(1 + \tfrac{1}{2} A_\times \sin 2\theta) \cos \omega t$$

International Symbol for Dangerous Gravitational Radiation



WARNING: GRAVITATIONAL RADIATION

Generating Gravitational Waves: Rough estimates

Waves at source will (at worst) have metric perturbations ~ 1 at the source's surface. Amplitude falls off as $1/r$.

First LIGO source: ~ 60 solar masses ≈ 10^5 m. Observed amplitude was 10^{-21}, so distance must have been about 10^{26} m = 10^{10} ly. Actual estimate was 1.3 Gy.

Sun as black hole eaten by another black hole: GM ≈ 3000 m. Amplitude of wave at earth ~ $1/(1.5 \times 10^{11}/3000) \approx 2 \times 10^{-8}$.

Generating Gravitational Waves: "Small-weak-slow" approximation

1. The source is *small* compared to both the wave's wavelength and the distance to the observer.

2. The source is *weak* in that $|h_{\mu\nu}| \ll 1$ even at the source.

3. The source is *slow* in that parts of the source move with speeds $v << 1$.

"Weak" means that we can use the weak-field Einstein equation:

$$\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu} \quad \text{subject to the Lorenz condition } \partial_\mu H^{\mu\nu} = 0$$

For which we know the solutions are:

$$H^{\mu\nu}(t, \vec{R}) = 4G \int_{\text{src}} \frac{T^{\mu\nu}(t - s, \vec{r})\, dV}{s} \quad \text{where } s \equiv |\vec{R} - \vec{r}|$$

Generating Gravitational Waves: "Small-weak-slow" approximation

"Small" means that $R \approx s$, and that the retarded time $t - s \approx t - R$:

$$H^{\mu\nu}(t, \vec{R}) = \left[\frac{4G}{R} \int_{\text{src}} T^{\mu\nu}\, dV \right]_{\text{at } t-R}$$

Overview: (1) $H^{tt} = 4GM/R = \text{constant, and } H^{ti} = H^{it} = 0$

(2) $$\int_{\text{src}} T^{ij}\, dV = \frac{1}{2}\frac{d^2}{dt^2}\int_{\text{src}} T^{tt} x^i x^j\, dV = \frac{1}{2}\frac{d^2}{dt^2}\int_{\text{src}} \rho x^i x^j\, dV \equiv \frac{1}{2}\ddot{I}^{ij}$$

(3) $$\rlap{-}I^{\,ij} \equiv \int_{\text{src}} \rho(x^i x^j - \tfrac{1}{3}\eta^{ij} r^2)\, dV \quad \text{where } r^2 \equiv x^2 + y^2 + z^2$$

(4) $$H^{xx}_{TT} = \tfrac{1}{2}(H^{xx} - H^{yy}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I}^{xx} - \ddot{I}^{yy}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{\rlap{-}I}^{\,xx} - \ddot{\rlap{-}I}^{\,yy}) \equiv \frac{2G}{R}\ddot{\rlap{-}I}^{\,xx}_{TT}$$

$$H^{yy}_{TT} = \tfrac{1}{2}(H^{yy} - H^{xx}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{I}^{yy} - \ddot{I}^{xx}) = \frac{2G}{R}\tfrac{1}{2}(\ddot{\rlap{-}I}^{\,yy} - \ddot{\rlap{-}I}^{\,xx}) \equiv \frac{2G}{R}\ddot{\rlap{-}I}^{\,yy}_{TT}$$

$$H^{xy}_{TT} = \frac{2G}{R}\ddot{I}^{xy} = \frac{2G}{R}\ddot{\rlap{-}I}^{\,xy} \equiv \frac{2G}{R}\ddot{\rlap{-}I}^{\,xy}_{TT}$$

Gravitational Wave Energy

Energy conservation in GR is a tricky topic, but here is a commonly accepted trick for handling energy in gravitational waves.

Einstein equation is to first order in the metric perturbation:

$$-2G^{(1)}_{\mu\nu} = \Box^2 H_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Trick is to expand the left side to second-order in the perturbation

$$-2G^{(1)}_{\mu\nu} - 2G^{(2)}_{\mu\nu} = \Box^2 H_{\mu\nu} - 2G^{(2)}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Treat the 2nd-order part as a stress-energy tensor for gravitational waves:

$$\Box^2 H_{\mu\nu} = -16\pi G T_{\mu\nu} + 2G^{(2)}_{\mu\nu} = -16\pi G (T_{\mu\nu} + T^{GW}_{\mu\nu}) \quad \text{where} \quad T^{GW}_{\mu\nu} \equiv \frac{G^{(2)}_{\mu\nu}}{8\pi G}$$

Because of the Lorenz gauge condition $\partial_\mu H^{\mu\nu} = 0$ we have

$$\partial_\mu (T^{\mu\nu} + T^{\mu\nu}_{GW}) = 0$$

Gravitational Wave Energy

Problems:

1. $T^{\mu\nu}_{GW}$ is only a tensor with regard to Lorentz-type transformations
2. It is not even invariant under a gauge transformation to TT!
3. But it does work if we average over several wavelengths:

$$T^{GW}_{\mu\nu} \equiv \frac{\langle G^{(2)}_{\mu\nu} \rangle}{8\pi G}$$

Gravitational Wave Energy

Consider the case of a plus polarized wave. Define:

$$h_+(t,z) \equiv A_+ \cos(\omega t - \omega z) = h_{xx}^{TT} = -h_{yy}^{TT}$$

$$\dot{h}_+ \equiv \partial_t h_+ = -\partial_z h_+ = -A_+ \omega \sin(\omega t - \omega z)$$

$$\ddot{h}_+ \equiv \partial_t \partial_t h_+ = \partial_z \partial_z h_+ = -\partial_t \partial_z h_+ = -\partial_z \partial_t h_+ = -\omega^2 h_+$$

Use Diagonal Metric Worksheet with $A = D = 1$, $B = 1 + h_+$, $C = 1 - h_+$:

$$B_0 = C_3 = -C_0 = -B_3 = \dot{h}_+$$

$$B_{00} = B_{33} = C_{03} = -B_{03} = -C_{00} = -C_{33} = \ddot{h}_+$$

$$\begin{aligned} R_{tt} &= -\tfrac{1}{2B} B_{00} - \tfrac{1}{2C} C_{00} + \tfrac{1}{4B^2} B_0^2 + \tfrac{1}{4C^2} C_0^2 \\ &= -\frac{\ddot{h}_+}{2(1+h_+)} - \frac{-\ddot{h}_+}{2(1-h_+)} + \frac{\dot{h}_+^2}{4(1+h_+)^2} + \frac{\dot{h}_+^2}{4(1-h_+)^2} \end{aligned}$$

Gravitational Wave Energy

Use binomial approximation and drop terms beyond second order:

$$R_{tt} = -\frac{\ddot{h}_+}{2(1+h_+)} - \frac{-\ddot{h}_+}{2(1-h_+)} + \frac{\dot{h}_+^2}{4(1+h_+)^2} + \frac{\dot{h}_+^2}{4(1-h_+)^2}$$

$$= -\tfrac{1}{2}\ddot{h}_+(1-h_+) + \tfrac{1}{2}\ddot{h}_+(1+h_+) + \tfrac{1}{2}\dot{h}_+^2 = \ddot{h}_+ h_+ + \tfrac{1}{2}\dot{h}_+^2$$

Now average over several wavelengths:

$$\langle R_{tt}\rangle = \langle \ddot{h}_+ h_+ + \tfrac{1}{2}\dot{h}_+^2\rangle = \langle -A_+^2\omega^2\cos^2\theta + \tfrac{1}{2}A_+^2\omega^2\sin^2\theta\,\rangle$$

$$= -\omega^2 A_+^2\langle\cos^2\theta - \sin^2\theta - \tfrac{1}{2}\sin^2\theta\rangle = -\omega^2 A_+^2\langle\sin 2\theta\rangle - \tfrac{1}{2}\omega^2 A_+^2\langle\sin^2\theta\rangle$$

$$= 0 - \tfrac{1}{2}\langle\dot{h}_+\dot{h}_+\rangle$$

Similarly: $R_{zz} = R_{tt} = -R_{tz} = -R_{zt}$, and all other $R_{\mu\nu} = 0$.

Exercise

The Diagonal Metric Worksheet's expansion for R_{tz} is

$$R_{tz} = -\frac{1}{2B}B_{03} - \frac{1}{2C}C_{03} + \frac{1}{4B^2}B_0B_3 - \frac{1}{4C^2}C_0C_3$$
$$+ \frac{1}{4AB}A_3B_0 + \frac{1}{4AC}A_3C_0 + \frac{1}{4DB}D_0B_3 + \frac{1}{4DC}D_0C_3$$

Remember that

$$B_0 = C_3 = -C_0 = -B_3 = \dot{h}_+$$

$$B_{00} = B_{33} = C_{03} = -B_{03} = -C_{00} = -C_{33} = \ddot{h}_+$$

Show then that $-R_{tz} = R_{tt} = \ddot{h}_+h_+ + \frac{1}{2}\dot{h}_+^2$

Gravitational Wave Energy

This means that

$$R = g^{\mu\nu} R_{\mu\nu} = -(1 - h^{tt}) R_{tt} + (1 - h^{zz}) R_{zz} = -(1+0) R_{tt} + (1+0) R_{tt} = 0$$

So the effective energy density of a plus-polarized wave is:

$$T_{tt}^{GW} = -\frac{\langle G_{tt}^{(2)} \rangle}{8\pi G} = -\frac{\langle R_{tt}^{(2)} \rangle}{8\pi G} = +\frac{\langle \dot{h}_+ \dot{h}_+ \rangle}{16\pi G}$$

The effective energy density of a cross-polarized wave must be the same, so the total energy density is

$$T_{tt}^{GW} = \frac{1}{16\pi G} \langle \dot{h}_+ \dot{h}_+ + \dot{h}_\times \dot{h}_\times \rangle \qquad \text{or} \qquad T_{tt}^{GW} = \frac{1}{32\pi G} \langle \dot{h}_{jk}^{TT} \dot{h}_{TT}^{jk} \rangle$$

Gravitational wave energy flux is (**Exercise:** Why flux = density?)

$$T_{GW}^{tz} = -T_{tz}^{GW} = +\frac{\langle R_{tz}^{(2)} \rangle}{8\pi G} = -\frac{\langle R_{tt}^{(2)} \rangle}{8\pi G} = \frac{1}{32\pi G} \langle \dot{h}_{jk}^{TT} \dot{h}_{TT}^{jk} \rangle = T_{tt}^{GW} \quad (!)$$

Source Luminosities: TT gauge for arbitrary directions

The problem for calculating luminosities: we have a different TT gauge for every wave direction (and we have only done it for the *z* direction).

Conceptually, this is not difficult. For a wave in the $+z$ direction, we:

1. Project $H^{\mu\nu}$ onto the plane perpendicular to the wave's direction
2. Subtract half of the trace of the projected matrix from the two remaining diagonal elements of the projected matrix.

Exercise: Consider an arbitrary $A^{\mu\nu}$ matrix for a wave moving in the $+x$ direction. Use the above steps to determine the transverse-traceless version of this matrix for that direction.

Source Luminosities: TT gauge for arbitrary directions

The solution for doing this for arbitrary directions is to express these operations in 3-tensor form that will give the correct components in any (rotated) coordinate system.

It turns out that this tensor projection operator projects a vector on the plane perpendicular to a unit vector $\vec{n}$: $P^i_j \equiv \delta^i_j - n^i n_j$

When $\vec{n}$ is in the +*z* direction, this becomes

$$P^i_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since a second-rank tensor ought to behave like the tensor product of two vectors, the projection of 3-tensor ought to be $P^i_m P^j_n I^{mn}$.

Source Luminosities: TT gauge for arbitrary directions

The trace of the projected matrix ought to be

$$I = \eta_{lk}(P^l_m P^k_n I^{mn}) = P_{mk} P^k_n I^{mn} = (\eta_{mk} - n_m n_k)(\delta^k_n - n^k n_m) I^{mn}$$
$$= (\eta_{mn} - n_m n_n - n_m n_n + n_m n_k n^k n_n) I^{mn} = (\eta_{mn} - n_m n_n) I^{mn} = P_{mn} I^{mn}$$

So the complete transformation ought to be:

$$I^{jk}_{TT} = (P^i_m P^j_n - \tfrac{1}{2} P^{ij} P_{mn}) I^{mn}$$

Energy flux of a gravitational wave: $\langle \dot{h}^{jk}_{TT} \dot{h}^{TT}_{jk} \rangle / 32\pi G$

and we also know that: $h^{ij}_{TT} = (2GM/R) \ddot{I\!\!\!-}^{ij}_{TT}$

So flux in a given direction is: $\text{flux} = \dfrac{G}{8\pi R^2} \langle \dddot{I\!\!\!-}^{ij}_{TT} \dddot{I\!\!\!-}^{TT}_{ij} \rangle$

Using the above (and lots of work) yields

$$\text{flux in } \vec{n}\text{-direction} = \frac{G}{16\pi R^2} \langle 2 \dddot{I\!\!\!-}^{ij} \dddot{I\!\!\!-}_{ij} - 4 n^i n^j \dddot{I\!\!\!-}_i{}^m \dddot{I\!\!\!-}_{mj} + n^i n^j n^m n^n \dddot{I\!\!\!-}_{ij} \dddot{I\!\!\!-}_{mn} \rangle$$

Source Luminosities: Integrating over all directions

To get the total luminosity, we integrate over all directions:

$$-\frac{dE}{dt} = \frac{G}{16\pi R^2}\int_0^{\pi}\left(\int_0^{2\pi}\langle \dddot{I}^{\,ij}_{TT}\,\dddot{I}^{\,TT}_{ij}\rangle\; d\phi\right) R^2 \sin\theta\, d\theta$$

$$= \frac{G}{16\pi}\int_0^{\pi}\left(\int_0^{2\pi}\langle 2\dddot{I}^{\,ij}\dddot{I}_{ij} - 4n^i n^j \dddot{I}_i{}^{m}\dddot{I}_{mj} + n^i n^j n^m n^n \dddot{I}_{ij}\dddot{I}_{mn}\rangle\, d\phi\right)\sin\theta\, d\theta$$

The quadrupole moment tensor does not depend on the wave direction:

$$-\frac{dE}{dt} = \frac{2G}{16\pi}\langle \dddot{I}^{\,ij}\dddot{I}_{ij}\rangle\int_0^{\pi}\left(\int_0^{2\pi} d\phi\right)\sin\theta\, d\theta - \frac{4G}{16\pi}\langle \dddot{I}_i{}^{m}\dddot{I}_{mj}\rangle\int_0^{\pi}\left(\int_0^{2\pi} n^i n^j\, d\phi\right)\sin\theta\, d\theta$$

$$+ \frac{G}{16\pi}\langle \dddot{I}_{ij}\dddot{I}_{mn}\rangle\int_0^{\pi}\left(\int_0^{2\pi} n^i n^j n^m n^n\, d\phi\right)\sin\theta\, d\theta$$

Each integral is just a number that depends on the index values.
When all is done:

$$-\frac{dE}{dt} = \frac{G}{5}\langle \dddot{I}^{\,ij}\dddot{I}_{ij}\rangle \qquad \textbf{Important!}$$

Gravitational Waves from Binaries: Fundamentals

Model: point masses m_1 and $m_2 > m_1$ separated by a fixed distance D, orbiting in the xy plane. We have:

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) D \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) D$$

$$x_1 = r_1 \cos\omega t = \frac{m_2 D}{m_1 + m_2}\cos\omega t \qquad \text{and} \qquad y_1 = r_1 \sin\omega t = \frac{m_2 D}{m_1 + m_2}\sin\omega t$$

$$x_2 = -r_2 \cos\omega t = -\frac{m_1 D}{m_1 + m_2}\cos\omega t \qquad \text{and} \qquad y_2 = -r_2 \sin\omega t = -\frac{m_1 D}{m_1 + m_2}\sin\omega t$$

So components of the reduced quadrupole moment tensor are:

$$\rlap{-}I^{xx} = \int_{\text{src}} \rho(x^2 - \tfrac{1}{3}\eta^{xx} r^2)\, dV = m_1(x_1^2 - \tfrac{1}{3}r_1^2) + m_2(x_2^2 - \tfrac{1}{3}r_2^2) = (m_1 r_1^2 + m_2 r_2^2)(\cos^2\omega t - \tfrac{1}{3})$$

$$\rlap{-}I^{xy} = \int_{\text{src}} \rho(xy - \tfrac{1}{3}\eta^{xy} r^2)\, dV = m_1(x_1 y_1 - 0) + m_2(x_2 y_2 - 0) = (m_1 r_1^2 + m_2 r_2^2)\cos\omega t \sin\omega t$$

Similarly, $\rlap{-}I^{yy} = (m_1 r_1^2 + m_2 r_2^2)(\sin^2\omega t - \frac{1}{3})$, $\rlap{-}I^{zz} = \frac{1}{3}(m_1 r_1^2 + m_2 r_2^2)$ and all other $\rlap{-}I^{ij} = 0$.

Gravitational Waves from Binaries: Fundamentals

We can simplify using double-angle formulas and the definitions

$$\eta \equiv \frac{m_1 m_2}{(m_1+m_2)^2} \quad \text{and} \quad M \equiv m_1 + m_2$$

Note also that

$$\begin{aligned} m_1 r_1^2 + m_2 r_2^2 &= m_1 \left(\frac{m_2 D}{m_1+m_2}\right)^2 + m_2 \left(\frac{m_1 D}{m_1+m_2}\right)^2 \\ &= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2} D^2 = \frac{m_1 m_2 D^2 (m_1+m_2)}{(m_1+m_2)^2} = \eta M D^2 \end{aligned}$$

So our reduced quadrupole moment tensor becomes

$$I\!\!\!-^{ij} = \tfrac{1}{2} M \eta D^2 \begin{bmatrix} \frac{1}{3} + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \frac{1}{3} - \cos 2\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

Gravitational Waves from Binaries: The Gravitational Waves

The reduced quadrupole moment tensor is

$$\rlap{-}I^{ij} = \tfrac{1}{2} M \eta D^2 \begin{bmatrix} \frac{1}{3} + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \frac{1}{3} - \cos 2\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

Its double time derivative is:

$$\ddot{\rlap{-}I}^{ij} = -2M\eta D^2 \omega^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is already in TT format for waves in the $+z$ direction. The wave is

$$h^{ij}_{TT} = H^{ij}_{TT} = \frac{2G}{R} \ddot{\rlap{-}I}^{ij}_{TT} = -\frac{4GM\eta D^2\omega^2}{R} \begin{bmatrix} \cos 2\omega(t-R) & \sin 2\omega(t-R) & 0 \\ \sin 2\omega(t-R) & -\cos 2\omega(t-R) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Gravitational Waves from Binaries: The Gravitational Waves

To find the waves radiated in another direction, use

$$\ddot{I}^{jk}_{TT} = (P^i_m P^j_n - \tfrac{1}{2} P^{ij} P_{mn}) \ddot{I}^{jk} \quad \text{with} \quad P^i_j = \delta^i_j - n^i n_j$$

or just calculate by eye: for the *x* direction:

$$h^{ij}_{TT} = \frac{2GM\eta D^2\omega^2}{R} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos 2\omega(t-R) & 0 \\ 0 & 0 & -\cos 2\omega(t-R) \end{bmatrix}$$

Exercise: verify this by applying the steps:

(1) Project $\ddot{I}^{ij}$ given below onto the plane perpendicular to the *x* direction

(2) Calculate the trace of the remaining components

(3) Subtract half the trace from each unaffected diagonal element

(4) Evaluate the wave at the retarded time.

$$\ddot{I}^{ij} = -2M\eta D^2\omega^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Gravitational Waves from Binaries: Total Luminosity

The binary's gravitational wave luminosity is:

$$-\frac{dE}{dt} = \frac{G}{5}\langle \dddot{I\!\!\!-}^{ij} \dddot{I\!\!\!-}_{ij}\rangle = \frac{32(GM)^2\eta^2 D^4\omega^6}{5G}$$

Exercise: We can derive this pretty easily from the equation below:

$$\ddot{I\!\!\!-}^{ij} = -2M\eta D^2\omega^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Do it. In particular, where does the 32 come from?

Gravitational Waves from Binaries: Effects of the Energy Loss

What effect does this have on the system itself? Newton's second law:

$$\frac{Gm_1m_2}{D^2} = \frac{m_1v_1^2}{r_1} \quad \Rightarrow \quad \frac{Gm_2}{D^2} = r_1\left(\frac{v_1}{r_1}\right)^2 = \frac{m_2D}{M}\omega^2 \quad \Rightarrow \quad D^3 = \frac{GM}{\omega^2}$$

Use this to eliminate D:

$$-\frac{dE}{dt} = \frac{32(GM)^2\eta\,\omega^6}{5G}\left(\frac{GM}{\omega^2}\right)^{4/3} = \frac{32\eta^2}{5G}(GM\omega)^{10/3}$$

Energy comes at the expense of orbital energy, which is:

$$E = -\frac{Gm_1m_2}{2D} = -\frac{G(\eta M^2)\omega^{2/3}}{2(GM)^{1/3}} = -\tfrac{1}{2}M(GM\omega)^{2/3}\eta$$

Our circular orbit approximation is not good if spiraling-in happens too fast!

Gravitational Waves from Binaries: Effects of the Energy Loss

One way of quantifying what "too fast" means: calculate dT/dt. Note

$$E = -\tfrac{1}{2}M(GM\omega)^{2/3}\eta \quad \Rightarrow \quad dE = -\tfrac{1}{3}M(GM)^{2/3}\omega^{-1/3}\eta\, d\omega$$

$$\Rightarrow \quad \frac{d\omega}{dE} = -\frac{3\,\omega^{1/3}}{M(GM)^{2/3}\eta}$$

$$\frac{dT}{dt} = \frac{dT}{d\omega}\frac{d\omega}{dE}\frac{dE}{dt} = \left(-\frac{2\pi}{\omega^2}\right)\left(-\frac{3\,\omega^{1/3}}{M(GM)^{2/3}\eta}\right)\left(-\frac{32\eta^2}{5G}(GM\omega)^{10/3}\right)$$

$$= \frac{192\pi\eta}{5}(GM\omega)^{5/3}$$

$$\frac{d\omega}{dt} = \frac{d\omega}{dE}\frac{dE}{dt} = \frac{96\eta}{5}(GM)^{5/3}\omega^{11/3} = \frac{96}{5}(G\mathcal{M})^{5/3}\omega^{11/3}$$

where $\mathcal{M} \equiv \eta^{3/5}M = \left(\frac{m_1 m_2}{[m_1+m_2]^2}\right)^{3/5}[m_1+m_2] = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$

"chirp mass"

Gravitational Waves from Binaries: Numbers for HM Cancri

HM Cancri (RX JO806.3+1527) consists of white dwarfs with masses of 0.55 and 0.27 solar masses (M = 1200 m, η = (0.149/0.822 = 0.22) with period 321.5 s = 9.6 × 10^{10} m. Distance ≈ 16,000 ly = 1.5 × 10^{20} m.
For this system $GM\omega$ = 7.9 × 10^{-8}. Orientation: nearly face on.

$$A_+ \approx \frac{4GM\eta D^2\omega^2}{R} = \frac{4GM\eta}{R}(GM\omega)^{2/3} = 1.3 \times 10^{-22}$$

$$-\frac{dE}{dt} = \frac{32\eta^2}{5G}(GM\omega)^{10/3} = 2.4 \times 10^{28}\ \mathrm{W}$$

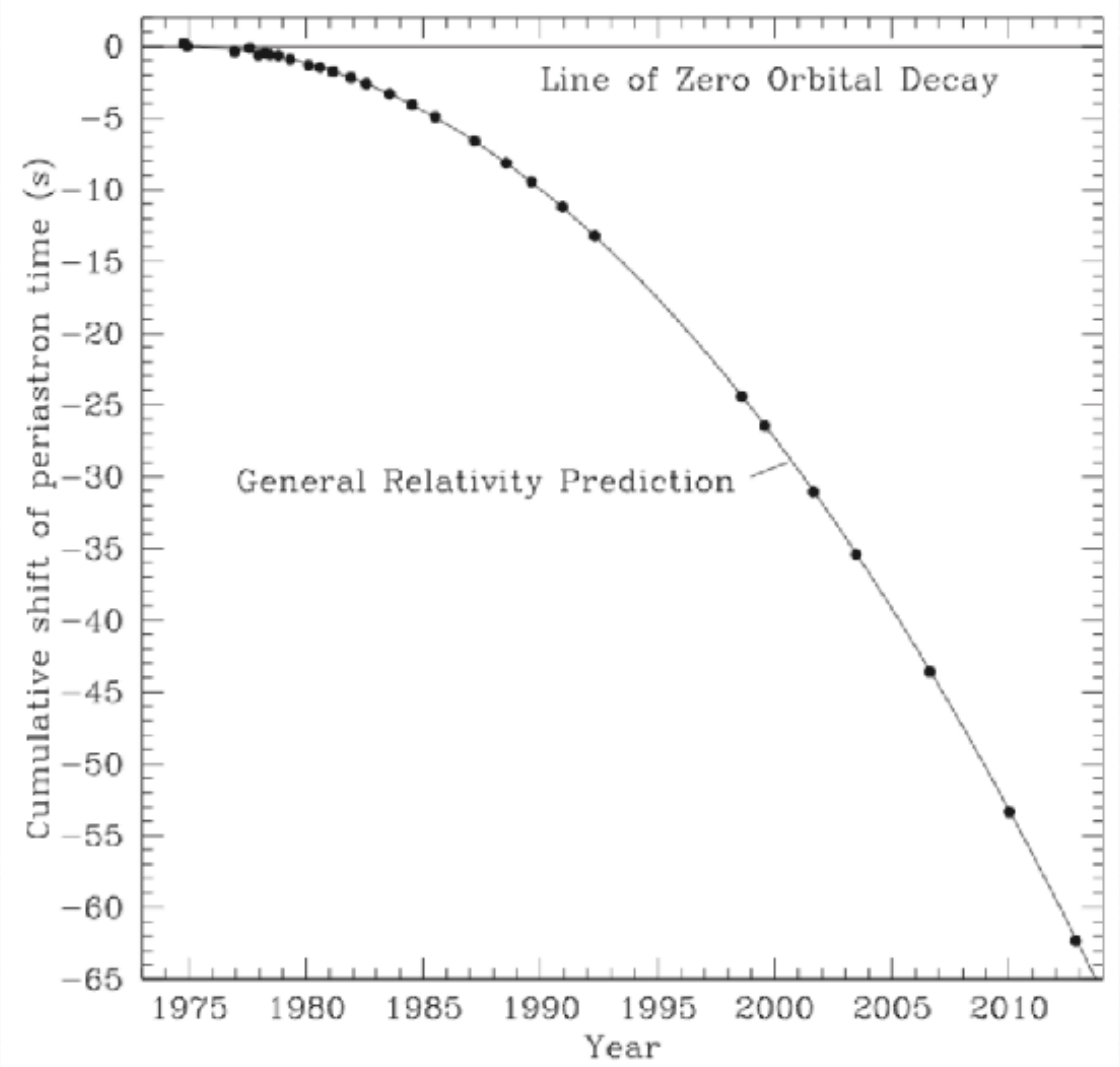
$$\frac{dT}{dt} = \frac{192\pi\eta}{5}(GM\omega)^{5/3} = -3.9 \times 10^{-11}$$

This is one of the best "LISA verification binaries" known.

Exercise: Verify the power using G = 7.426 × 10^{-28} m/kg to get the power in kg/m, then convert to watts by multiplying by the appropriate power of c (which is what power?)

Gravitational Waves from Binaries: Pre-2015 Evidence for GWs

Hulse-Taylor Binary (PSR B1913+16) (Graph from Weisberg and Huang, *The Astrophysical Journal*, 829:55, 2016 September 20)



Gravitational Waves from Binaries: Post-Newtonian calculations

A more sophisticated model keeps track of corrections to the Newtonian model as a power series in orders of v^2. The resulting expression for the waveform looks like

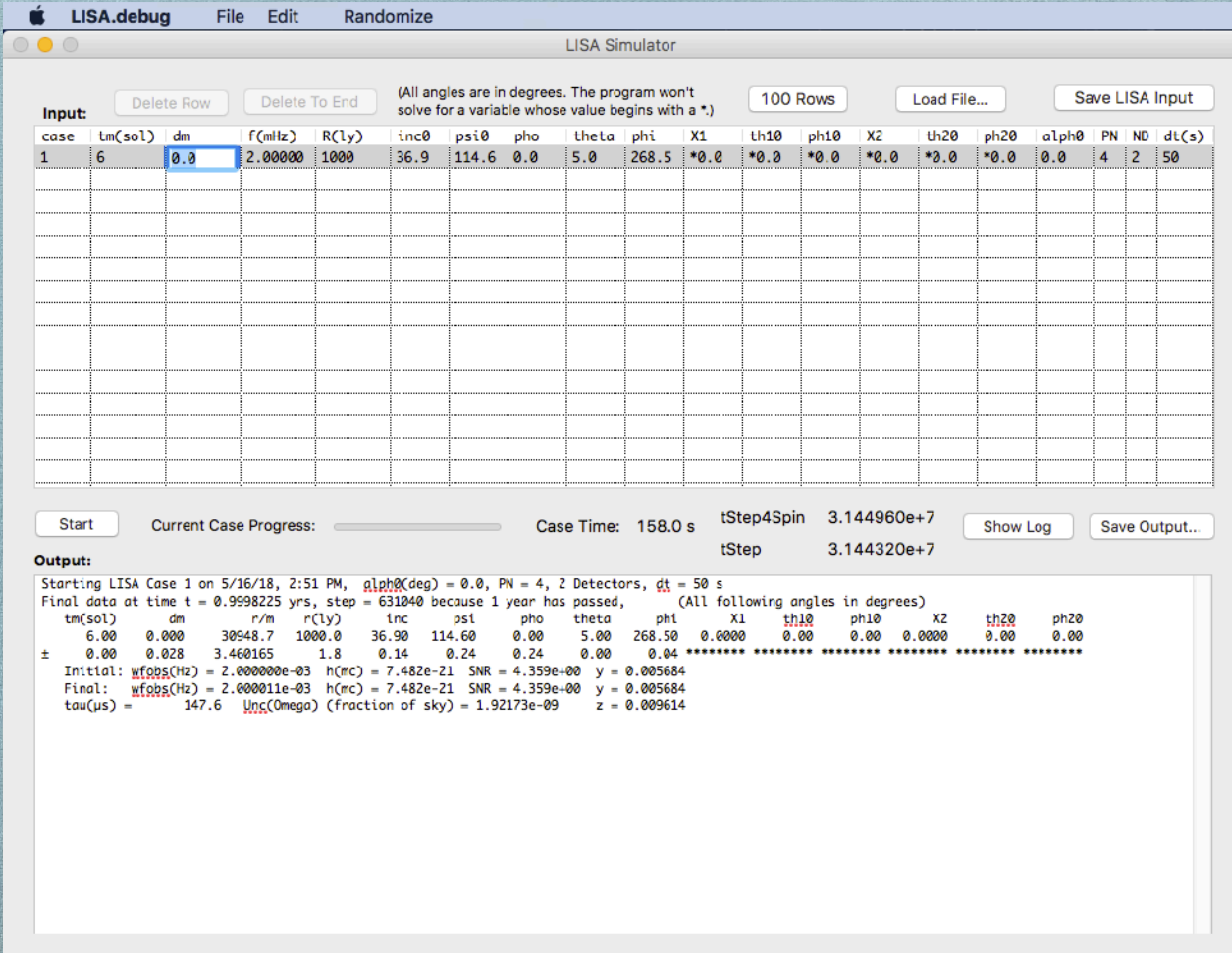
$$h_+ = \frac{2GM\eta}{R}(GM\omega)^{2/3}(B_+^{(0)} + y^{1/2}B_+^{(1/2)} + y^1 B_+^{(1)} + y^{3/2}B_+^{(3/2)} + y^2 B_+^{(2)})$$

with $y \equiv (GM\omega)^{2/3} \approx GM/D \approx v^2$ and

$$B_+^{(0)} = -(1+\cos^2 i)\cos 2\Psi,$$

$$B_+^{(1/2)} = -\frac{\sin i}{8}\frac{\delta m}{M}\left[(5+\cos^2 i)\cos\Psi + 9(1+\cos^2 i)\cos 3\Psi\right]$$

We would say that this expression is accurate to “second post-Newtonian order” or 2PN.


LISA.debug
File
Edit
Randomize
LISA Simulator
Input:
Delete Row
Delete To End
(All angles are in degrees. The program won't solve for a variable whose value begins with a *.)
100 Rows
Load File...
Save LISA Input
case | tm(sol) | dm | f(mHz) | R(ly) | inc0 | psi0 | pho | theta | phi | X1 | th10 | ph10 | X2 | th20 | ph20 | alph0 | PN | ND | dt(s)
1 | 6 | 0.0 | 2.00000 | 1000 | 36.9 | 114.6 | 0.0 | 5.0 | 268.5 | *0.0 | *0.0 | *0.0 | *0.0 | *0.0 | *0.0 | 0.0 | 4 | 2 | 50
Start
Current Case Progress:
Case Time: 158.0 s
tStep4Spin 3.144960e+7
tStep 3.144320e+7
Show Log
Save Output...
Output:
Starting LISA Case 1 on 5/16/18, 2:51 PM, alph0(deg) = 0.0, PN = 4, 2 Detectors, dt = 50 s
Final data at time t = 0.9998225 yrs, step = 631040 because 1 year has passed, (All following angles in degrees)
tm(sol) dm r/m r(ly) inc psi pho theta phi X1 th10 ph10 X2 th20 ph20
6.00 0.000 30948.7 1000.0 36.90 114.60 0.00 5.00 268.50 0.0000 0.00 0.00 0.0000 0.00 0.00
± 0.00 0.028 3.460165 1.8 0.14 0.24 0.24 0.00 0.04 ******** ******** ******** ******** ******** ********
Initial: wfobs(Hz) = 2.000000e-03 h(mc) = 7.482e-21 SNR = 4.359e+00 y = 0.005684
Final: wfobs(Hz) = 2.000011e-03 h(mc) = 7.482e-21 SNR = 4.359e+00 y = 0.005684
tau(μs) = 147.6 Unc(Omega) (fraction of sky) = 1.92173e-09 z = 0.009614

Thank you for your kind attention!

Ich bin der Mann!