The Problem with Photons. Since the proper time τ along a photon worldline is zero, we cannot *directly* apply the geodesic equation 10.1 or any of the equations of motion derived from it in chapter 10 to photons. However (as discussed in chapter 8), ordinary particles become more and more photon-like as their mass m becomes negligible compared to their energy. To find equations of motion for photons in Schwarzschild spacetime, we will therefore use the equations in chapter 10 in combinations that remain well-defined in the limit that $m \to 0$.

The Impact Parameter *b*. With this in mind, let us define

$$b = \frac{\ell}{e} = \frac{r^2(d\phi/d\tau)}{(1-2GM/r)(dt/d\tau)} = r^2\left(1-\frac{2GM}{r}\right)^{-1}\frac{d\phi}{dt} \tag{12.1}$$

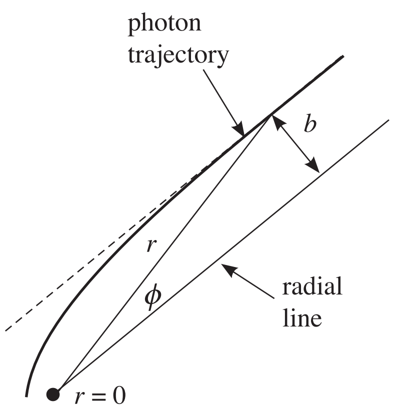

Figure 12.1: This figure illustrates the meaning of the impact parameter b.

This is a conserved quantity for any geodesic, and it remains perfectly well-defined as we take $m \to 0$. The quantity b corresponds to perpendicular distance between the particle's trajectory at very large r and the radial line initially parallel to that trajectory, as shown in figure 12.1 (see Box 12.1). In classical physics, we call this distance the trajectory's **impact parameter**, and we will continue that usage here.

Note that in flat space, this expression would simply be $b = r^2 d\phi/dt$.

The Equation of Radial Motion for a Photon. According to the Schwarzschild metric equation, the coordinate differences dt, dr, and $d\phi$ between two infinitesimally-separated events along an equatorial *photon* worldline will be related by

$$0 = ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\phi^2 \tag{12.2}$$

If we divide both sides by $(1-2GM/r)\,dt^2$, use equation 12.1, and rearrange things a bit (see Box 12.2), we get

$$1 = \left(1-\frac{2GM}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 + \frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right) \tag{12.3}$$

Equations 12.1 and 12.3 provide a complete set of equations of motion for the photon in that, given b, we could (in principle) solve equation 12.3 for $r(t)$ and then substitute this into equation 12.1 and to find $\phi(t)$. This parameterizes the photon's motion not in terms of proper time τ (as we would with an ordinary particle) but in terms of the Schwarzschild time coordinate t.

We can put this equation into a more evocative form if we divide both sides of equation 12.3 by b^2 to get

$$\frac{1}{b^2} = \left[\frac{1}{b}\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{dt}\right]^2 + \frac{1}{r^2}\left(1-\frac{2GM}{r}\right) \tag{12.4}$$

This equation has the form of a conservation of energy equation, where $1/b^2$ plays the role of the conserved energy, the first quantity on the right side is a complicated quantity we might consider a "radial kinetic energy", and the last term plays the role of an effective potential energy. With the help of this equation, we can read characteristics of a photon's motion off a graph of potential energy graph like the one shown in figure 12.2 (see Box 12.3) for a discussion of the features of this effective potential energy function). In particular, note that for $b > \sqrt{27}\,GM$ (that is, for $1/b^2 < 1/27[GM]^2$), a photon coming in from infinity will rebound to infinity, but for impact parameters less than this critical value, the photon will spiral in to $r = 0$. Note also that photons have a possible (unstable) circular orbit at $r = 3GM$.

The equivalent equation of motion for flat space (see Box 12.4) is

$$\frac{1}{b^2} = \left[\frac{1}{b}\frac{dr}{dt}\right]^2 + \frac{1}{r^2} \qquad \text{(flat space)} \tag{12.5}$$

Note that in this case, the effective potential energy function $1/r^2$ has no peak, but rather increases to infinity as r becomes small. This means that a photon coming in from infinity will always go back out to infinity, as one would expect in flat space.